Summary of Fractal Exponent Calculation

I. If an object is made up of N identical copies of itself, shrunk by a factor of S, then the fractal dimension *D* of the object is:

$$D = \frac{log\ N}{Log\ (1/S)} \tag{1}$$

The base of the logarithm can be 10 (common log), *e* (natural log), or whatever you want. For the Sierpinski triangle, N = 3 and S = ½. So $D = 1.58496\ldots$

II. Divider method of computing the fractal dimension *D* of a coastline, or other line or boundary. Assume we have a ruler of length δ and that we can cover the entire curve by a number N_δ of ruler lengths. For a smooth curve (not fractal), we expect that the length of the line *L* is a constant when we take the limit as $\delta \rightarrow 0$:

$$\lim_{\delta\rightarrow 0} \delta\, N_\delta = L(\delta) = \text{constant} \tag{2}$$

For a fractal curve, which is not smooth in the sense of having unique slope at each point, this doesn't work. As the ruler length δ becomes smaller and smaller, we typically find that

$$\lim_{\delta\rightarrow 0} \delta\, N_\delta = L(\delta) \rightarrow \infty \tag{3}$$

So the answer here is that we *postulate* or *assume* a relationship involving a fractal dimension *D* such that

$$\lim_{\delta\rightarrow 0} \delta^D\, N_\delta = B = \text{constant} \tag{4}$$

Then to find the fractal dimension of the line, we measure its length $\delta\, N_\delta \equiv L$ at 2 different length scales, δ_1 and δ_2. We then use equation (4):

$$\delta_1{}^D\, N_{\delta_1} = B = \delta_2{}^D\, N_{\delta_2} \tag{5}$$

Rearranging terms, we have:

$$\left(\frac{\delta_2}{\delta_1}\right)^D = \frac{N_{\delta_1}}{N_{\delta_2}} \tag{6}$$

To solve this for *D*, we need to use some properties of logarithms.

<u>Brief digression into logarithms</u>

Lets look at some properties of logarithms. If $y = x^D$ then:

$$x = 10^{\log_{10} x} \tag{7}$$

So $y = x^D$ becomes:

$$y = x^D = (10^{\log_{10} x})^D = 10^{D \log_{10} x} \tag{8}$$

Note that we could also have written (8) as:

$$y = x^D = (e^{\log_e x})^D = e^{D \log_e x} \tag{9}$$

using the base of natural logarithms *e*. Alternately, we could write (9) in slightly different notation as:

$$y = x^D = (e^{ln\, x})^D = e^{D\ ln\, x} \tag{10}$$

We can solve (8) as:

$$\log_{10} y = D \log_{10} x \quad \text{or} \quad D = \frac{\log_{10} y}{\log_{10} x} \tag{11}$$

We can solve (9) as:

$$\log_e y = D \log_e x \quad \text{or} \quad D = \frac{\log_e y}{\log_e x} \tag{12}$$

and we can solve (10) as:

$$\ln y = D \ln x \quad \text{or} \quad D = \frac{\ln y}{\ln x} \tag{13}$$

<u>Let's now return to solving for *D* in equation (6).</u>

Since $(\frac{\delta_2}{\delta_1})^D = \frac{N_{\delta_1}}{N_{\delta_2}}$, we can use, for example, (11):

$$D = \frac{\log_{10}\left(\frac{N_{\delta_1}}{N_{\delta_2}}\right)}{\log_{10}\left(\frac{\delta_2}{\delta_1}\right)} \tag{14}$$

Alternatively, we could use equation (12):

$$D = \frac{\log_{e}\left(\frac{N_{\delta_1}}{N_{\delta_2}}\right)}{\log_{e}\left(\frac{\delta_2}{\delta_1}\right)} \tag{15}$$

So it really doesn't matter which base of the logarithm we use, it could be base 10 or base *e* or for that matter any other base.

III. Box covering method of computing fractal dimension of a curve or shape. This is basically the same as the divider method, but is often easier to implement. Here what you do is to cover the shape with boxes of side length δ. The number of such boxes is N_δ. That is, N_δ represents the number of boxes in which any part of the curve appears. We then use two different box sizes δ_1 and δ_2 and find 2 different numbers N_{δ_1} and N_{δ_2}. To find *D* we then use (15).

A final note. If you apply these methods to compute the fractal dimension of a normal smooth line or shape, you will find that the fractal dimension *D* that you compute is the normal "embedding" dimension *d* of the space the space in which it appears. For example, the embedding dimension of the space in which the Sierpinski triangle is embedded is $d = 2$. So if you use these methods to compute the fractal dimension of an ordinary right triangle, you will find $D = d = 2$.