Fourier Series & The Fourier Transform

What is the Fourier Transform?

Fourier Cosine Series for even functions and Sine Series for odd functions

The continuous limit: the Fourier transform (and its inverse)

The spectrum

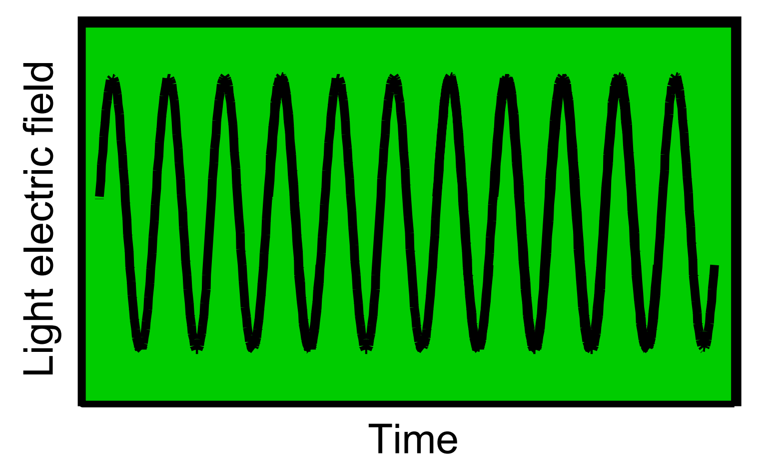
Some examples and theorems

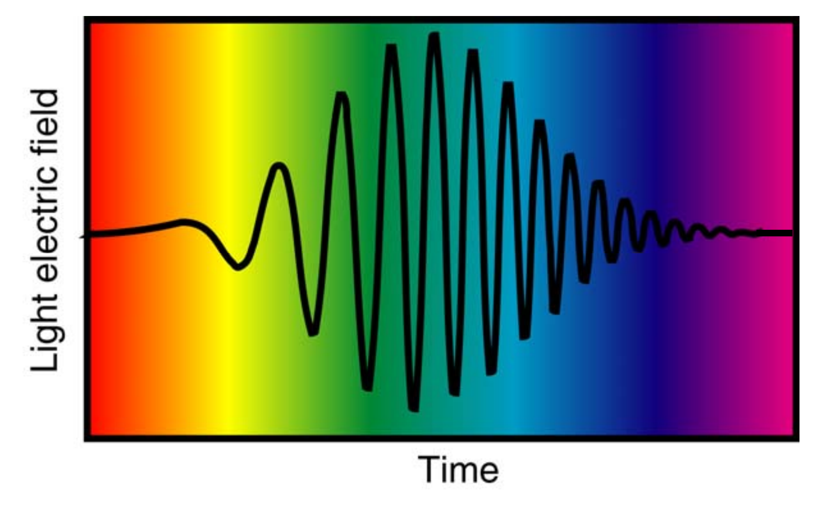
$$f(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(\omega)\exp(i\omega t)\,d\omega \qquad F(\omega) = \int_{-\infty}^{\infty} f(t)\exp(-i\omega t)\,dt$$

What do we want from the Fourier Transform?

We desire a measure of the frequencies present in a wave.
This will lead to a definition of the term, the “spectrum.”

Plane waves have only one frequency, ω.





This light wave has many frequencies. And the frequency increases in time (from red to blue).

It will be nice if our measure also tells us when each frequency occurs.

Lord Kelvin on Fourier’s theorem

Fourier’s theorem is not only one of the most beautiful results of modern analysis, but it may be said to furnish an indispensable instrument in the treatment of nearly every recondite question in modern physics.

Lord Kelvin

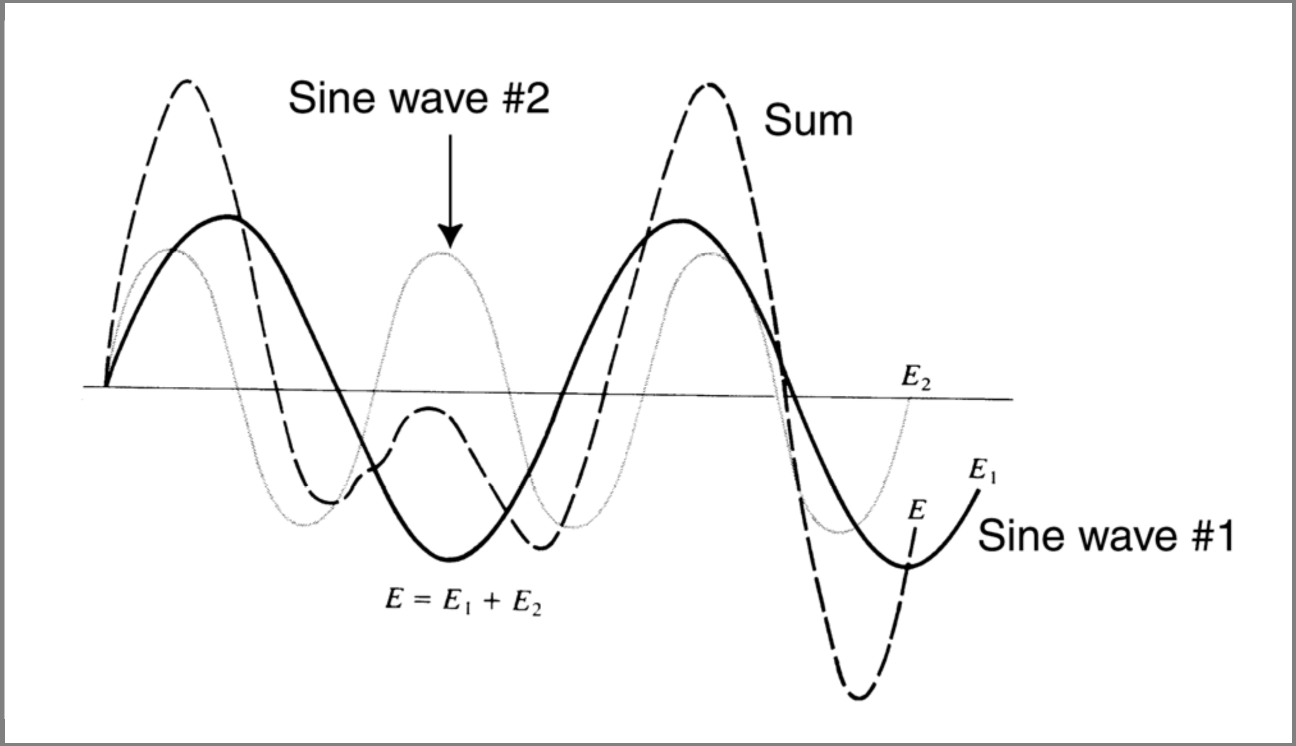
Joseph Fourier, our hero

Joseph Fourier, 21 March 1768–16 May 1830. (*By permission of the Bibliothèque Municipale de Grenoble.*)

Fourier was obsessed with the physics of heat and developed the Fourier series and transform to model heat-flow problems.

Anharmonic waves are sums of sinusoids.

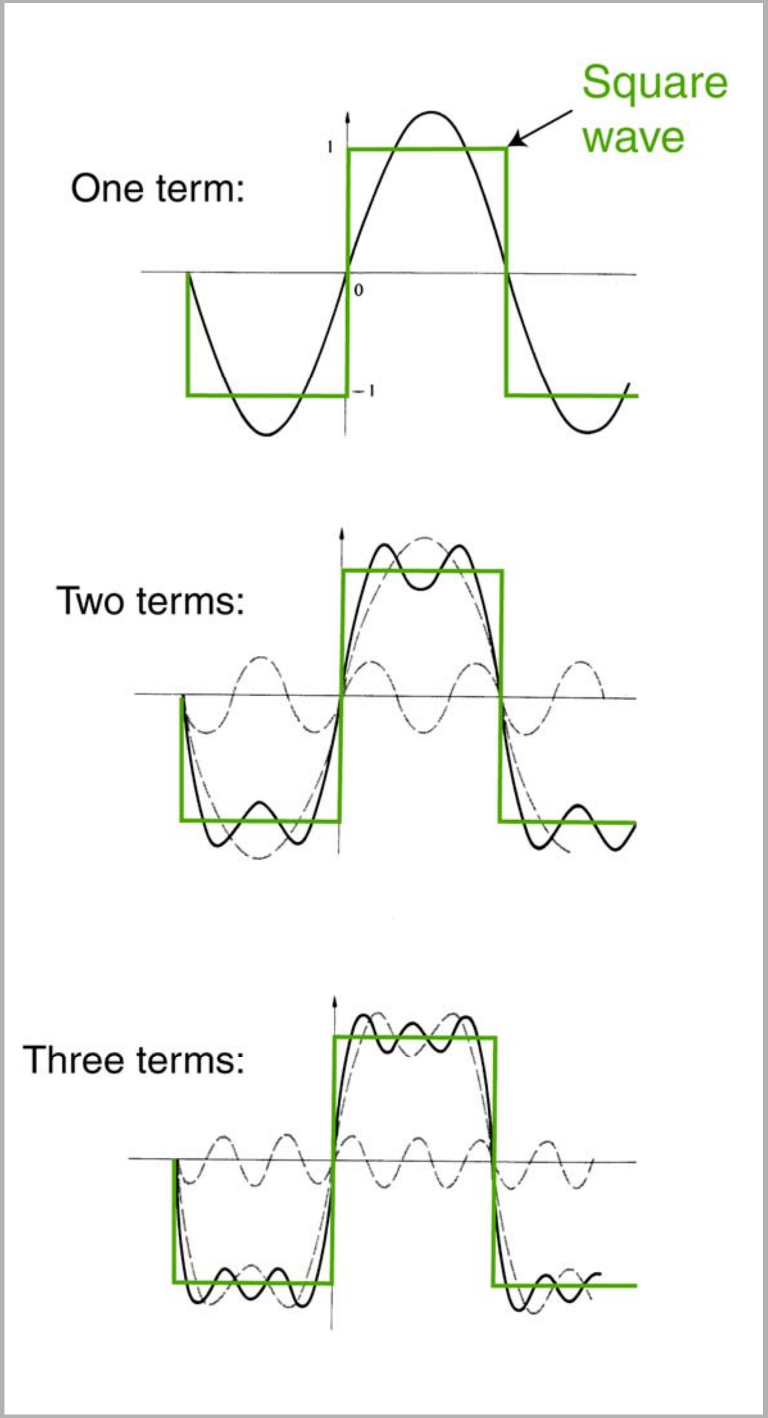
Consider the sum of two sine waves (i.e., harmonic waves) of different frequencies:



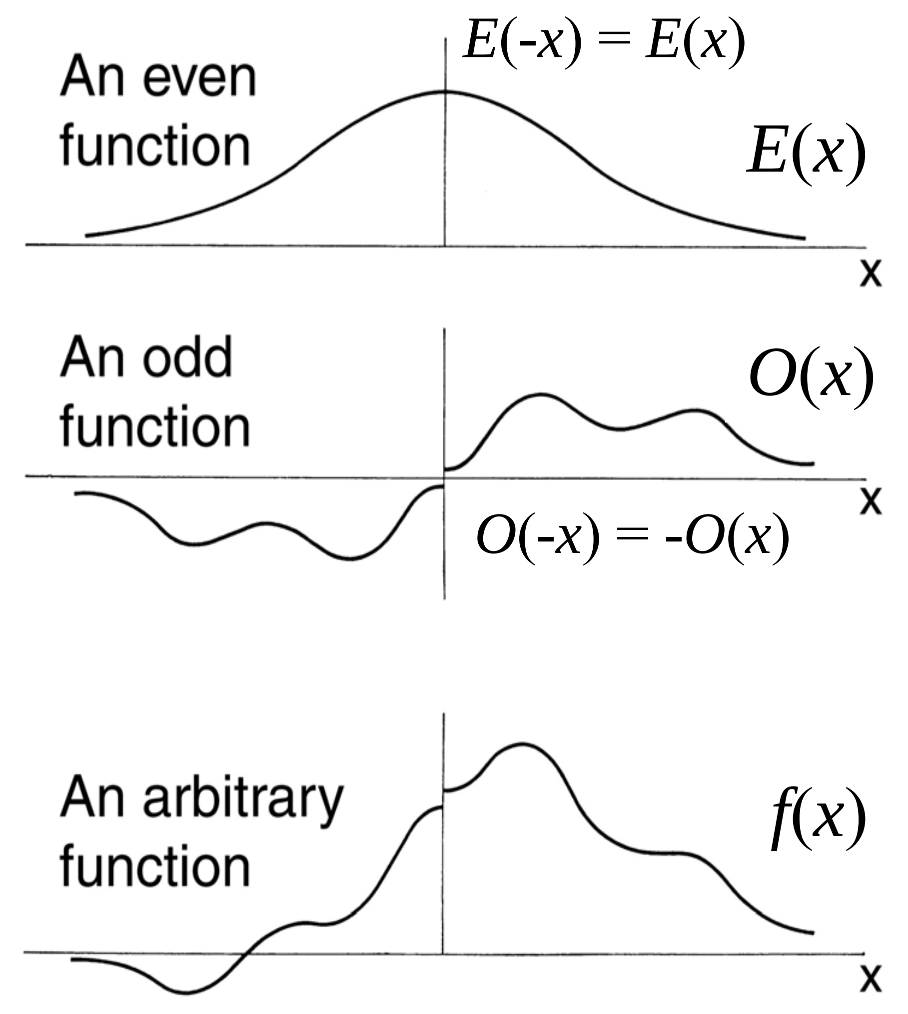
The resulting wave is periodic, but not harmonic.
Essentially all waves are anharmonic.

Fourier decomposing functions

Here, we write a
square wave as
a sum of sine waves.



Any function can be written as the sum of an even and an odd function



Let $f(x)$ be any function.

$$E(x) \equiv [f(x) + f(-x)]/2$$

$$O(x) \equiv [f(x) - f(-x)]/2$$

$$\Downarrow$$

$$f(x) = E(x) + O(x)$$

Fourier Cosine Series

Because $\cos(mt)$ is an even function (for all m), we can write an even function, $f(t)$, as:

$$f(t) \;=\; \frac{1}{\pi}\sum_{m=0}^{\infty} F_m \cos(mt)$$

where the set $\{F_m;\, m = 0, 1, \ldots\}$ is a set of coefficients that define the series.

And where we'll only worry about the function $f(t)$ over the interval $(-\pi,\pi)$.

The Kronecker delta function

$$\delta_{m,n} \equiv \begin{cases} 1 \text{ if } m = n \\ 0 \text{ if } m \neq n \end{cases}$$

Finding the coefficients, F_m, in a Fourier Cosine Series

Fourier Cosine Series: $$f(t) = \frac{1}{\pi}\sum_{m=0}^{\infty} F_m \cos(mt)$$

To find F_m, multiply each side by $\cos(m't)$, where m' is another integer, and integrate:

$$\int_{-\pi}^{\pi} f(t)\cos(m't)\,dt = \frac{1}{\pi}\sum_{m=0}^{\infty}\int_{-\pi}^{\pi} F_m \cos(mt)\cos(m't)\,dt$$

But: $$\int_{-\pi}^{\pi} \cos(mt)\cos(m't)\,dt = \begin{cases} \pi & if\ m = m' \\ 0 & if\ m \neq m' \end{cases} \equiv \pi\,\delta_{m,m'}$$

So: $$\int_{-\pi}^{\pi} f(t)\cos(m't)\,dt = \frac{1}{\pi}\sum_{m=0}^{\infty} F_m\,\pi\,\delta_{m,m'}$$ ← only the $m' = m$ term contributes

Dropping the ' from the m:

$$\boxed{F_m = \int_{-\pi}^{\pi} f(t)\cos(mt)\,dt}$$

← yields the coefficients for any $f(t)$!

Fourier Sine Series

Because $\sin(mt)$ is an odd function (for all m), we can write any odd function, $f(t)$, as:

$$f(t) = \frac{1}{\pi} \sum_{m=0}^{\infty} F'_m \sin(mt)$$

where the set $\{F'_m;\ m = 0, 1, \ldots\}$ is a set of coefficients that define the series.

where we'll only worry about the function $f(t)$ over the interval $(-\pi,\pi)$.

Finding the coefficients, F'_m, in a Fourier Sine Series

Fourier Sine Series:
$$f(t) = \frac{1}{\pi}\sum_{m=0}^{\infty} F'_m \sin(mt)$$

To find F_m, multiply each side by $\sin(m't)$, where m' is another integer, and integrate:
$$\int_{-\pi}^{\pi} f(t)\sin(m't)\,dt = \frac{1}{\pi}\sum_{m=0}^{\infty}\int_{-\pi}^{\pi} F'_m \sin(mt)\sin(m't)\,dt$$

But:
$$\int_{-\pi}^{\pi} \sin(mt)\sin(m't)\,dt = \begin{cases}\pi & if\ m = m' \\ 0 & if\ m \neq m'\end{cases} \equiv \pi\,\delta_{m,m'}$$

So:
$$\int_{-\pi}^{\pi} f(t)\sin(m't)\,dt = \frac{1}{\pi}\sum_{m=0}^{\infty} F'_m\,\pi\,\delta_{m,m'}$$
← only the $m' = m$ term contributes

Dropping the ‘ from the m:
$$\boxed{F'_m = \int_{-\pi}^{\pi} f(t)\sin(mt)\,dt}$$
← yields the coefficients for any $f(t)$!

Fourier Series

So if $f(t)$ is a general function, neither even nor odd, it can be written:

$$\boxed{f(t) = \frac{1}{\pi}\sum_{m=0}^{\infty} F_m \cos(mt) + \frac{1}{\pi}\sum_{m=0}^{\infty} F'_m \sin(mt)}$$

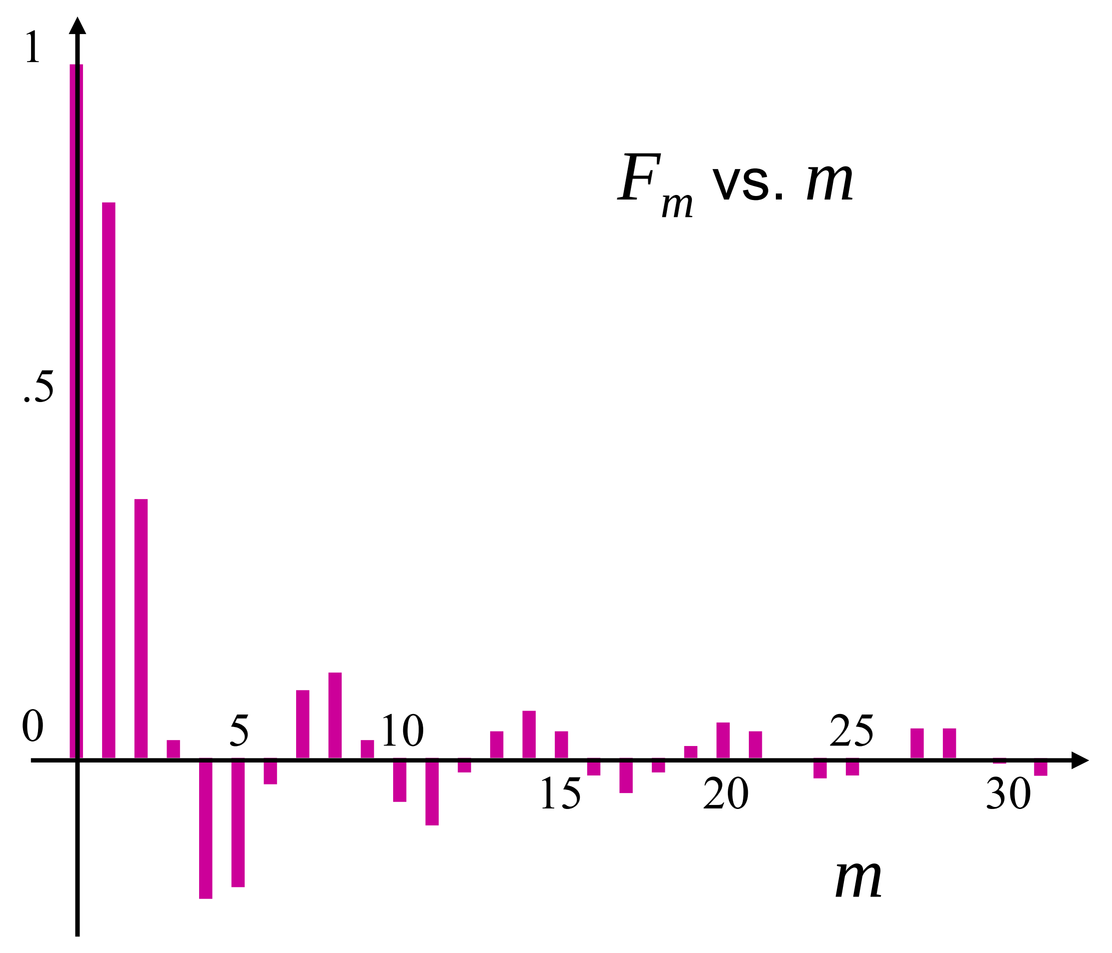
even component

odd component

where

$$F_m = \int f(t)\cos(mt)\,dt \quad \text{and} \quad F'_m = \int f(t)\sin(mt)\,dt$$

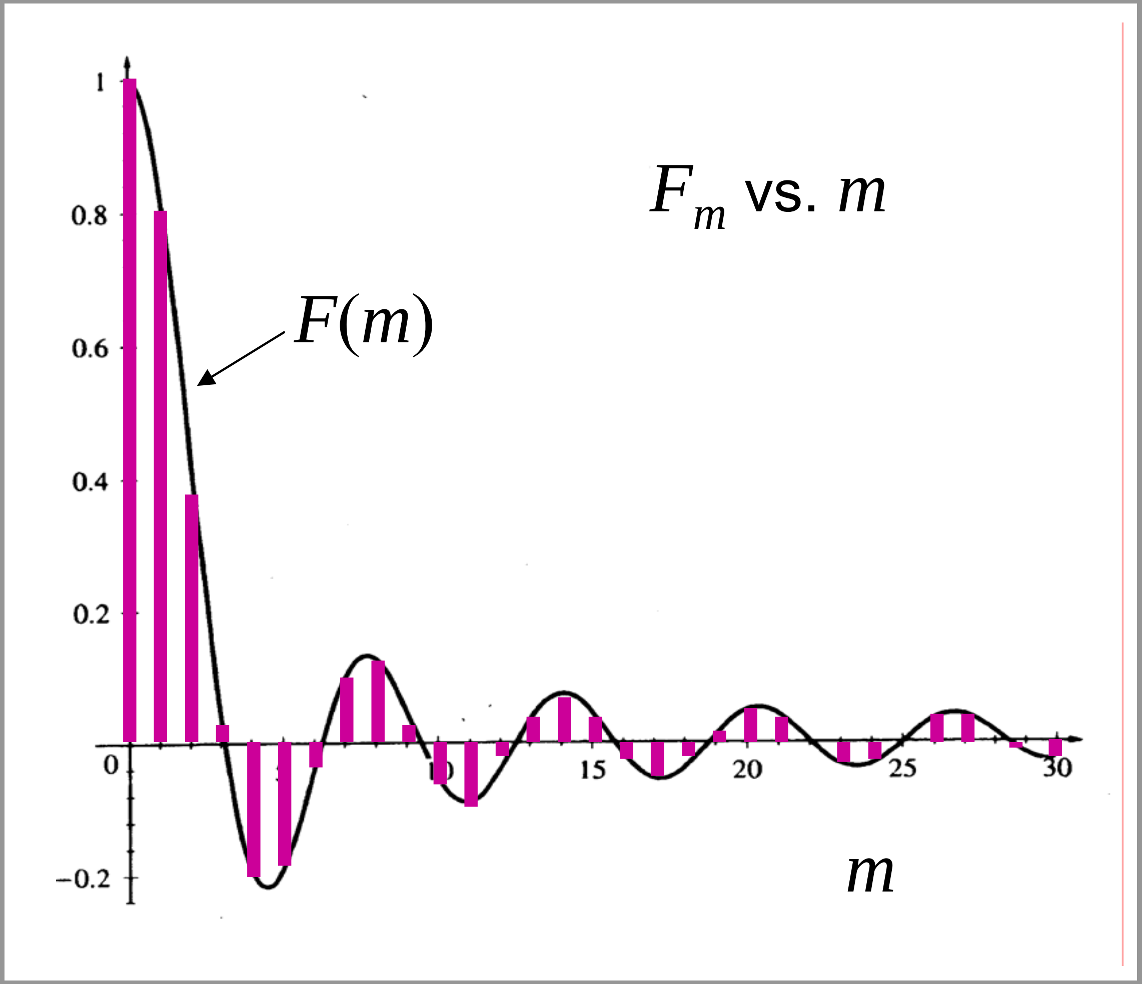
We can plot the coefficients of a Fourier Series



We really need two such plots, one for the cosine series and another for the sine series.

Discrete Fourier Series vs. Continuous Fourier Transform

Let the integer m become a real number and let the coefficients, F_m, become a function $F(m)$.



Again, we really need two such plots, one for the cosine series and another for the sine series.

The Fourier Transform

Consider the Fourier coefficients. Let's define a function $F(m)$ that incorporates both cosine and sine series coefficients, with the sine series distinguished by making it the imaginary component:

$$F(m) \equiv F_m - i\,F'_m = \int f(t)\cos(mt)\,dt \; - \; i \int f(t)\sin(mt)\,dt$$

Let's now allow $f(t)$ to range from $-\infty$ to ∞, so we'll have to integrate from $-\infty$ to ∞, and let's redefine m to be the "frequency," which we'll now call ω:

$$\boxed{F(\omega) = \int_{-\infty}^{\infty} f(t)\exp(-i\omega t)\,dt}$$

The Fourier Transform

$F(\omega)$ is called the Fourier Transform of $f(t)$. **It contains equivalent information to that in $f(t)$.** We say that $f(t)$ lives in the **time domain**, and $F(\omega)$ lives in the **frequency domain**. $F(\omega)$ is just another way of looking at a function or wave.

The Inverse Fourier Transform

The Fourier Transform takes us from $f(t)$ to $F(\omega)$.
How about going back?

Recall our formula for the Fourier Series of $f(t)$:

$$f(t) \quad = \quad \frac{1}{\pi}\sum_{m=0}^{\infty} F_m \cos(mt) \quad + \quad \frac{1}{\pi}\sum_{m=0}^{\infty} F_m^{'} \sin(mt)$$

Now transform the sums to integrals from –∞ to ∞, and again replace F_m with $F(\omega)$. Remembering the fact that we introduced a factor of i (and including a factor of 2 that just crops up), we have:

$$\boxed{f(t) \quad = \quad \frac{1}{2\pi}\int_{-\infty}^{\infty} F(\omega) \exp(i\omega t)\, d\omega}$$

Inverse Fourier Transform

Fourier Transform Notation

There are several ways to denote the Fourier transform of a function.

If the function is labeled by a lower-case letter, such as f, we can write:

$$f(t) \;\rightarrow\; F(\omega)$$

If the function is labeled by an upper-case letter, such as E, we can write:

$$E(t) \rightarrow \mathscr{F}\{E(t)\} \quad \text{or:} \quad E(t) \rightarrow \tilde{E}(\omega)$$

Sometimes, this symbol is used instead of the arrow: $\supset$

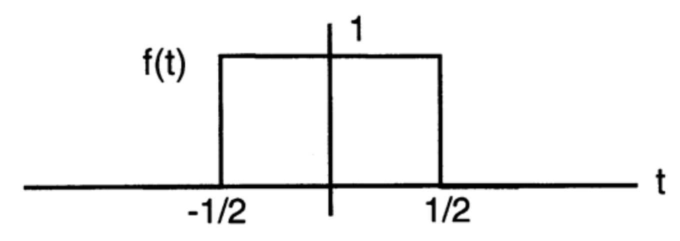
The Spectrum

We define the spectrum, $S(\omega)$, of a wave $E(t)$ to be:

$$\boxed{S(\omega) \equiv \left| \mathcal{F}\{E(t)\} \right|^2}$$

This is the measure of the frequencies present in a light wave.

Example: the Fourier Transform of a rectangle function: rect(t)

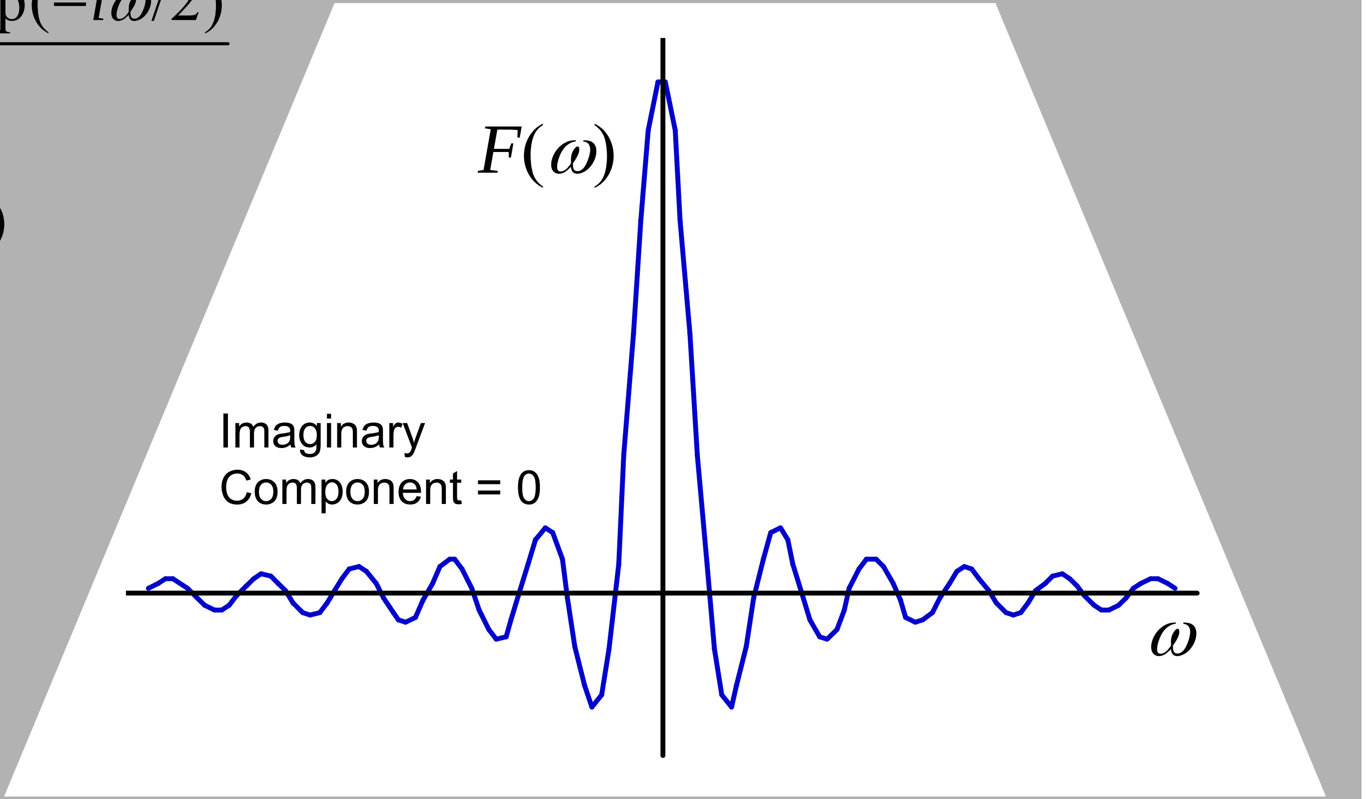


$$F(\omega) = \int_{-1/2}^{1/2} \exp(-i\omega t)dt = \frac{1}{-i\omega}[\exp(-i\omega t)]_{-1/2}^{1/2}$$

$$= \frac{1}{-i\omega}[\exp(-i\omega / 2) - \exp(i\omega/2)]$$

$$= \frac{1}{(\omega/2)} \frac{\exp(i\omega / 2) - \exp(-i\omega/2)}{2i}$$

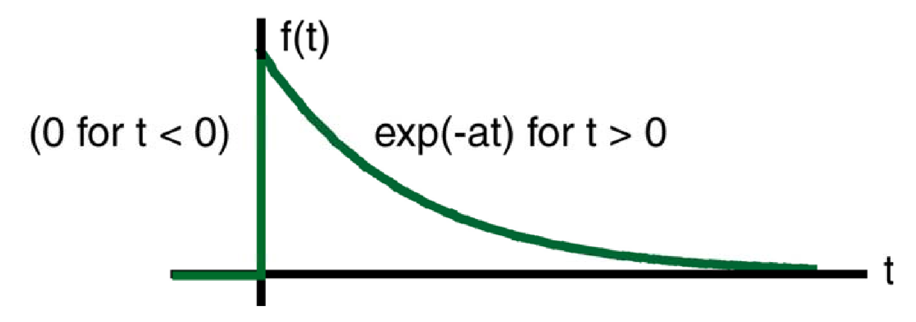
$$= \frac{\sin(\omega/2)}{(\omega/2)} \equiv \text{sinc}(\omega/2)$$



Imaginary Component = 0

$$\boxed{\text{F} \; \{\text{rect}(t)\} = \text{sinc}(\omega/2)}$$

Example: the Fourier Transform of a decaying exponential: exp(-*at*) (*t* > 0)



$$F(\omega) = \int_0^\infty \exp(-at)\exp(-i\omega t)dt$$

$$= \int_0^\infty \exp(-at - i\omega t)dt = \int_0^\infty \exp(-[a+i\omega]t)dt$$

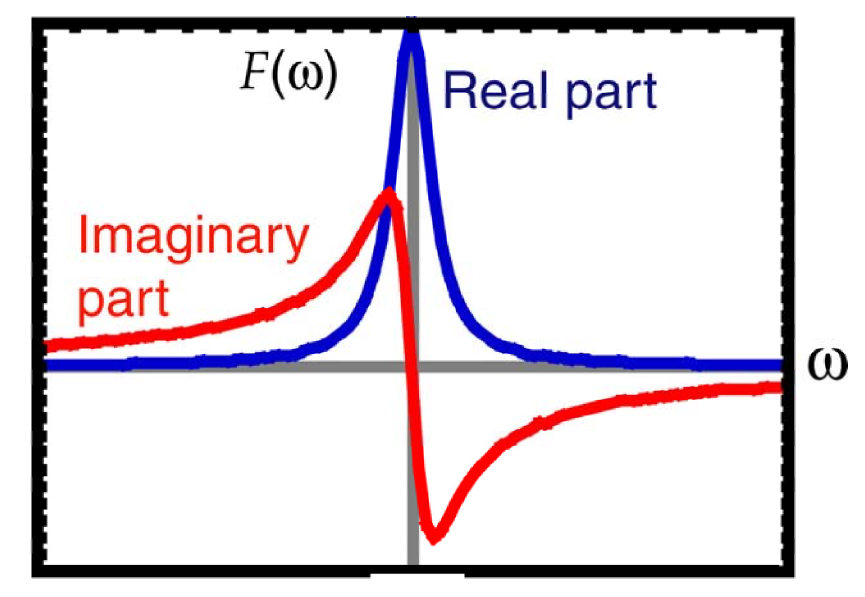
$$= \frac{-1}{a+i\omega}\exp(-[a+i\omega]t)\Big|_0^{+\infty} = \frac{-1}{a+i\omega}[\exp(-\infty) - \exp(0)]$$

$$= \frac{-1}{a+i\omega}[0-1]$$

$$= \frac{1}{a+i\omega}$$

$$\boxed{F(\omega) = -i\frac{1}{\omega - ia}}$$

A complex Lorentzian!

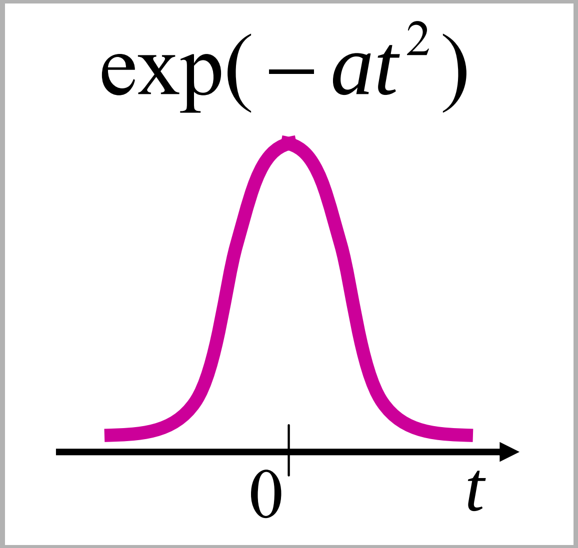


Example: the Fourier Transform of a Gaussian, $\exp(-at^2)$, is itself!

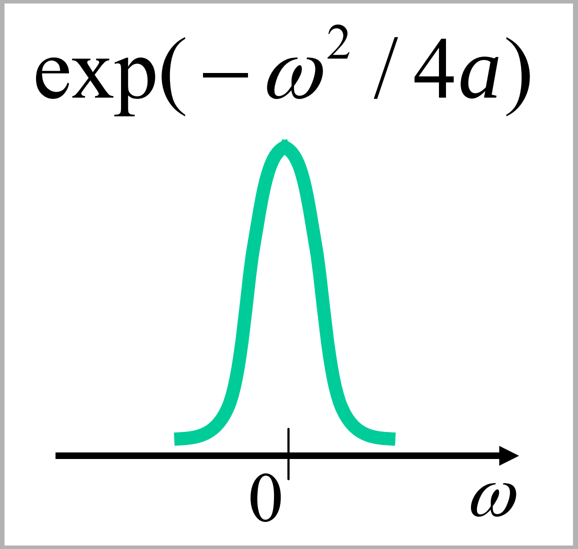
$$\mathrm{F}\ \{\exp(-at^2)\} = \int_{-\infty}^{\infty} \exp(-at^2)\exp(-i\omega t)\,dt$$

$$\boxed{\propto \exp(-\omega^2/4a)}$$

The details are a HW problem!



$\supset$



Fourier Transform Symmetry Properties

Expanding the Fourier transform of a function, *f(t):*

$$F(\omega) = \int_{-\infty}^{\infty} [\mathrm{Re}\{f(t)\} + i\,\mathrm{Im}\{f(t)\}]\;[\cos(\omega t) - i\sin(\omega t)]\,dt$$

Expanding more, noting that: $\int_{-\infty}^{\infty} O(t)\,dt = 0$ if $O(t)$ is an odd function

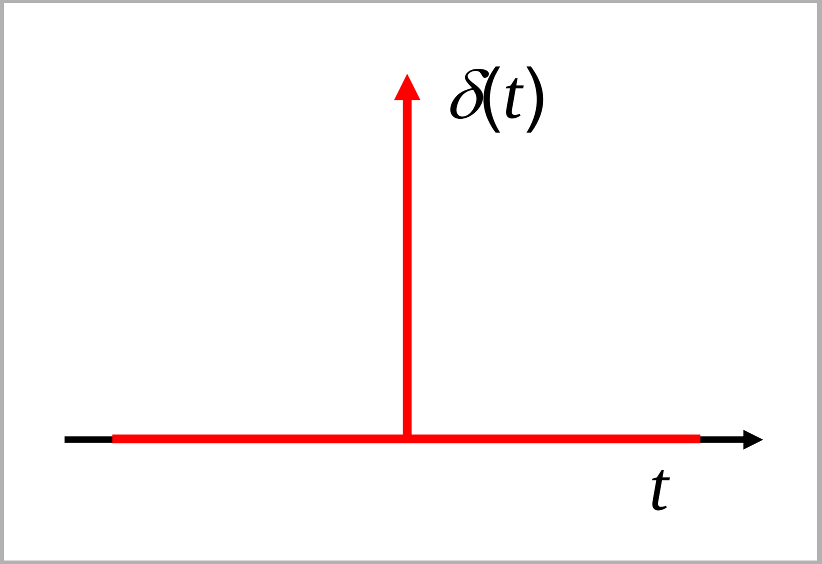
$$F(\omega) = \int_{-\infty}^{\infty} \overset{\substack{\text{= 0 if Re}\{f(t)\}\text{ is odd}\\ \downarrow}}{\mathrm{Re}\{f(t)\}\cos(\omega t)}\,dt + \int_{-\infty}^{\infty} \overset{\substack{\text{= 0 if Im}\{f(t)\}\text{ is even}\\ \downarrow}}{\mathrm{Im}\{f(t)\}\sin(\omega t)}\,dt \;\leftarrow \mathrm{Re}\{F(\omega)\}$$

$$+\; i\int_{-\infty}^{\infty} \underset{\substack{\uparrow\\ \text{Even functions of } \omega}}{\overset{\substack{\text{= 0 if Im}\{f(t)\}\text{ is odd}\\ \downarrow}}{\mathrm{Im}\{f(t)\}\cos(\omega t)}}\,dt \;-\; i\int_{-\infty}^{\infty} \underset{\substack{\uparrow\\ \text{Odd functions of } \omega}}{\overset{\substack{\text{= 0 if Re}\{f(t)\}\text{ is even}\\ \downarrow}}{\mathrm{Re}\{f(t)\}\sin(\omega t)}}\,dt \;\leftarrow \mathrm{Im}\{F(\omega)\}$$

The Dirac delta function

Unlike the Kronecker delta-function, which is a function of two integers, the Dirac delta function is a function of a real variable, t.

$$\boxed{\delta(t) \equiv \begin{cases} \infty \text{ if } t = 0 \\ 0 \text{ if } t \neq 0 \end{cases}}$$

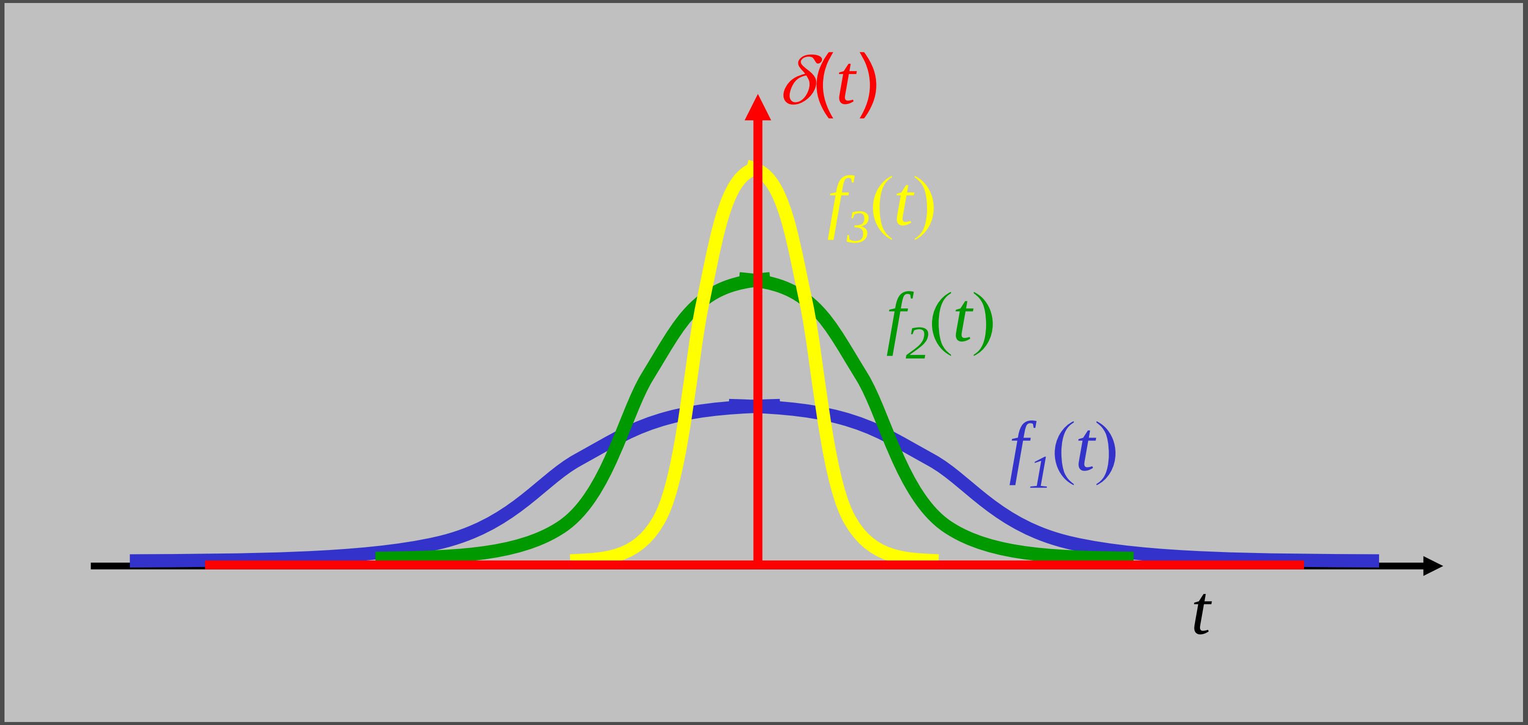


The Dirac delta function

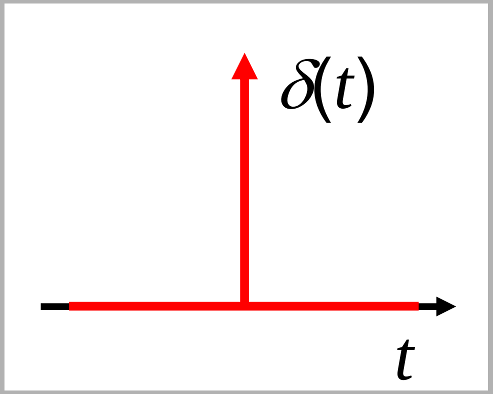
$$\delta(t) \equiv \begin{cases} \infty \text{ if } t = 0 \\ 0 \text{ if } t \neq 0 \end{cases}$$

It's best to think of the delta function as the limit of a series of peaked continuous functions.

$$f_m(t) = m \exp[-(mt)^2]/\sqrt{\pi}$$



Dirac δ–function Properties



$$\int_{-\infty}^{\infty} \delta(t)\, dt = 1$$

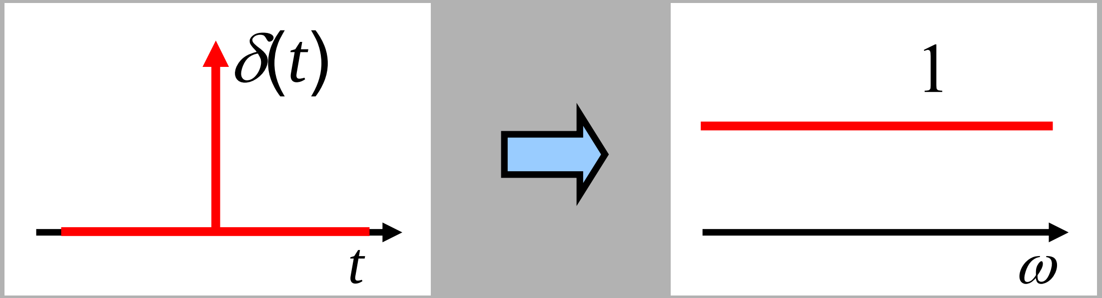
$$\int_{-\infty}^{\infty} \delta(t-a)\, f(t)\, dt = \int_{-\infty}^{\infty} \delta(t-a)\, f(a)\, dt = f(a)$$

$$\int_{-\infty}^{\infty} \exp(\pm i\omega t)\, dt = 2\pi\, \delta(\omega)$$

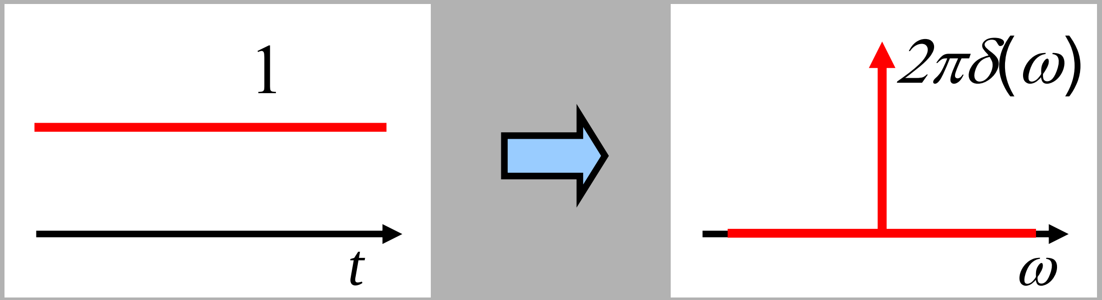
$$\int_{-\infty}^{\infty} \exp[\pm i(\omega-\omega')t]\ dt = 2\pi\, \delta(\omega-\omega')$$

The Fourier Transform of $\delta(t)$ is 1.

$$\int_{-\infty}^{\infty} \delta(t)\exp(-i\omega t)\,dt = \exp(-i\omega[0]) = 1$$



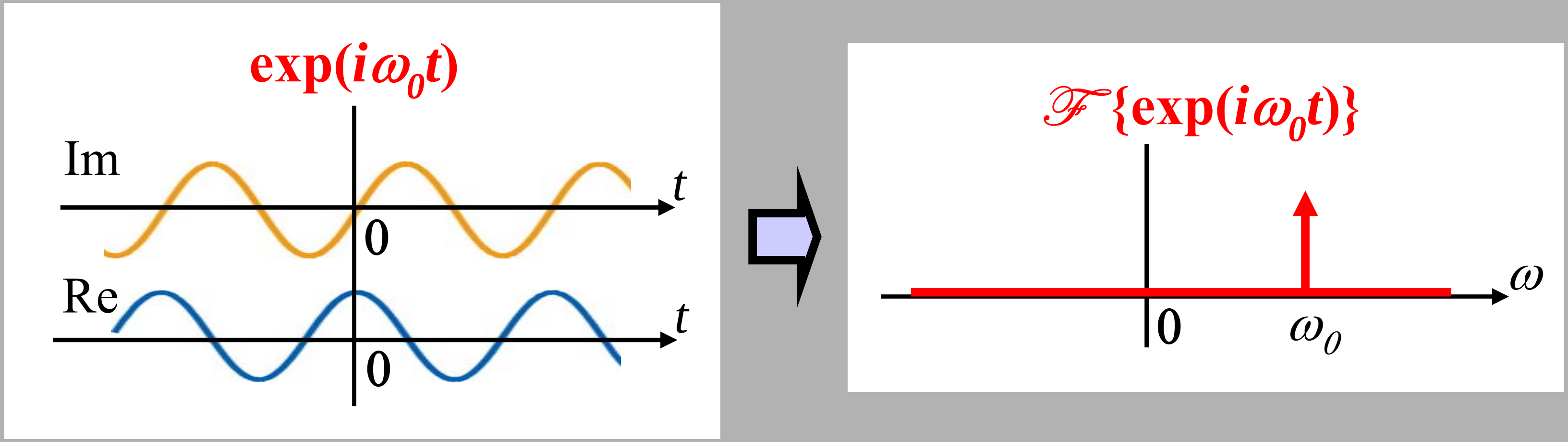
And the Fourier Transform of 1 is $2\pi\delta(\omega)$: $$\int_{-\infty}^{\infty} 1\exp(-i\omega t)\,dt = 2\pi\,\delta(\omega)$$



The Fourier transform of $\exp(i\omega_0 t)$

$$\mathrm{F}\ \left\{\exp(i\omega_0 t)\right\} = \int_{-\infty}^{\infty} \exp(i\omega_0\, t)\exp(-i\,\omega t)\,dt$$

$$= \int_{-\infty}^{\infty} \exp(-i[\omega - \omega_0]t)\,dt \ = 2\pi\,\delta(\omega - \omega_0)$$



The function $\exp(i\omega_0 t)$ is the essential component of Fourier analysis.

It is a pure frequency.

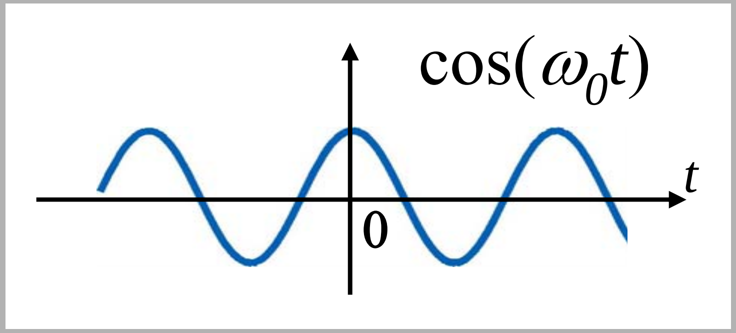
The Fourier transform of $\cos(\omega_0 t)$

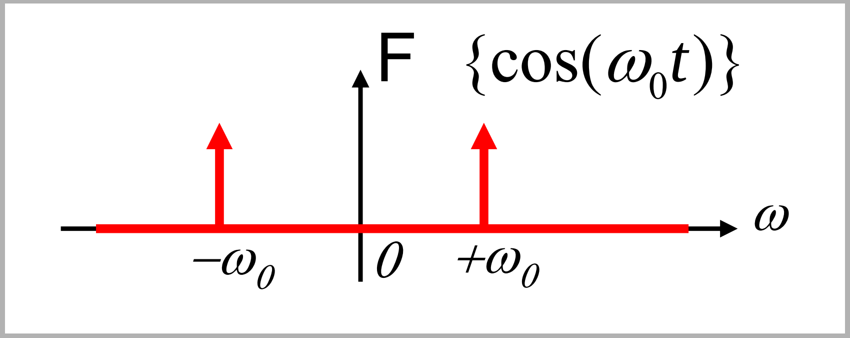
$$\mathcal{F}\{\cos(\omega_0 t)\} = \int_{-\infty}^{\infty} \cos(\omega_0 t)\exp(-i\,\omega t)\,dt$$

$$= \frac{1}{2}\int_{-\infty}^{\infty} \left[\exp(i\,\omega_0\,t) + \exp(-i\,\omega_0\,t)\right]\exp(-i\,\omega t)\,dt$$

$$= \frac{1}{2}\int_{-\infty}^{\infty} \exp(-i[\omega-\omega_0]t)\,dt \quad + \quad \frac{1}{2}\int_{-\infty}^{\infty} \exp(-i[\omega+\omega_0]t)\,dt$$

$$= \pi\,\delta(\omega-\omega_0) \quad + \quad \pi\,\delta(\omega+\omega_0)$$





The Modulation Theorem:
The Fourier Transform of $E(t)\cos(\omega_0 t)$

$$\text{F}\left\{E(t)\cos(\omega_0 t)\right\} = \int_{-\infty}^{\infty} E(t)\cos(\omega_0 t)\exp(-i\,\omega\,t)\,dt$$

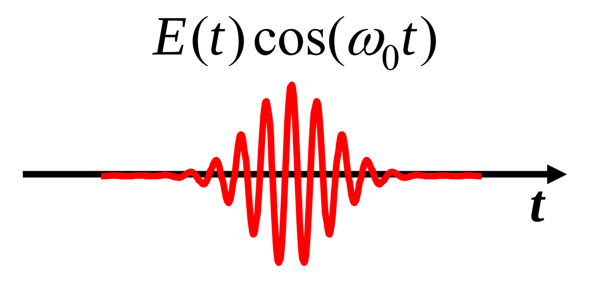
$$= \frac{1}{2}\int_{-\infty}^{\infty} E(t)\left[\exp(i\,\omega_0 t) + \exp(-i\,\omega_0 t)\right]\exp(-i\,\omega\,t)\,dt$$

$$= \frac{1}{2}\int_{-\infty}^{\infty} E(t)\exp(-i[\omega-\omega_0]t)\,dt \quad + \quad \frac{1}{2}\int_{-\infty}^{\infty} E(t)\exp(-i[\omega+\omega_0]t)\,dt$$

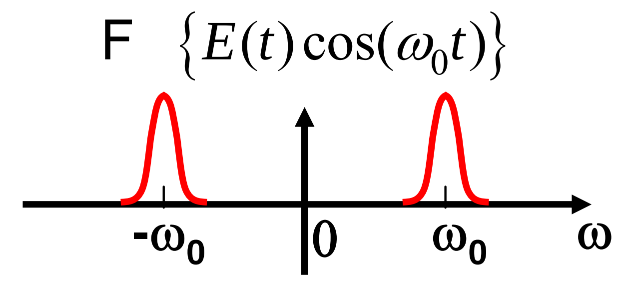
$$\boxed{\text{F}\left\{E(t)\cos(\omega_0 t)\right\} = \frac{1}{2}\tilde{E}(\omega-\omega_0) \quad + \quad \frac{1}{2}\tilde{E}(\omega+\omega_0)}$$

Example:

$E(t) = \exp(-t^2)$



$E(t)\cos(\omega_0 t)$



$\text{F}\left\{E(t)\cos(\omega_0 t)\right\}$

Scale Theorem

The Fourier transform
of a scaled function, $f(at)$:

$$\boxed{\mathcal{F}\ \{f(at)\} = F(\omega/a)\ /\ |a|}$$

Proof:

$$\mathcal{F}\ \{f(at)\} = \int_{-\infty}^{\infty} f(at)\exp(-i\omega t)\, dt$$

Assuming $a > 0$, change variables: $u = at$

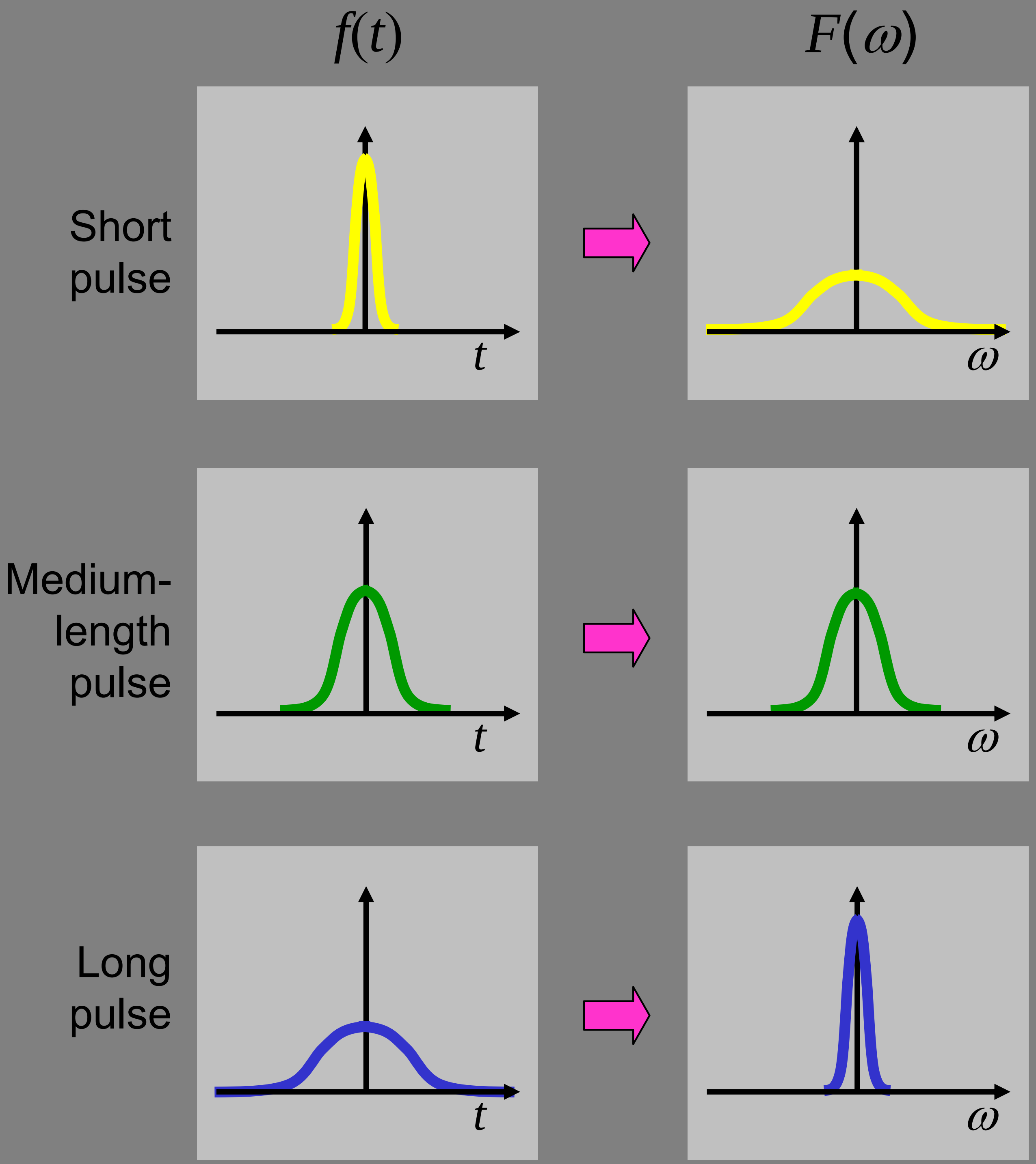
$$\mathcal{F}\ \{f(at)\} = \int_{-\infty}^{\infty} f(u)\exp(-i\omega[u/a])\, du\ /\ a$$

$$= \int_{-\infty}^{\infty} f(u)\exp(-i\ [\omega/a]\ u)\, du\ \ /\ \ a$$

$$= F(\omega/a)\ /\ a$$

If $a < 0$, the limits flip when we change variables, introducing a minus sign, hence the absolute value.

The Scale Theorem in action



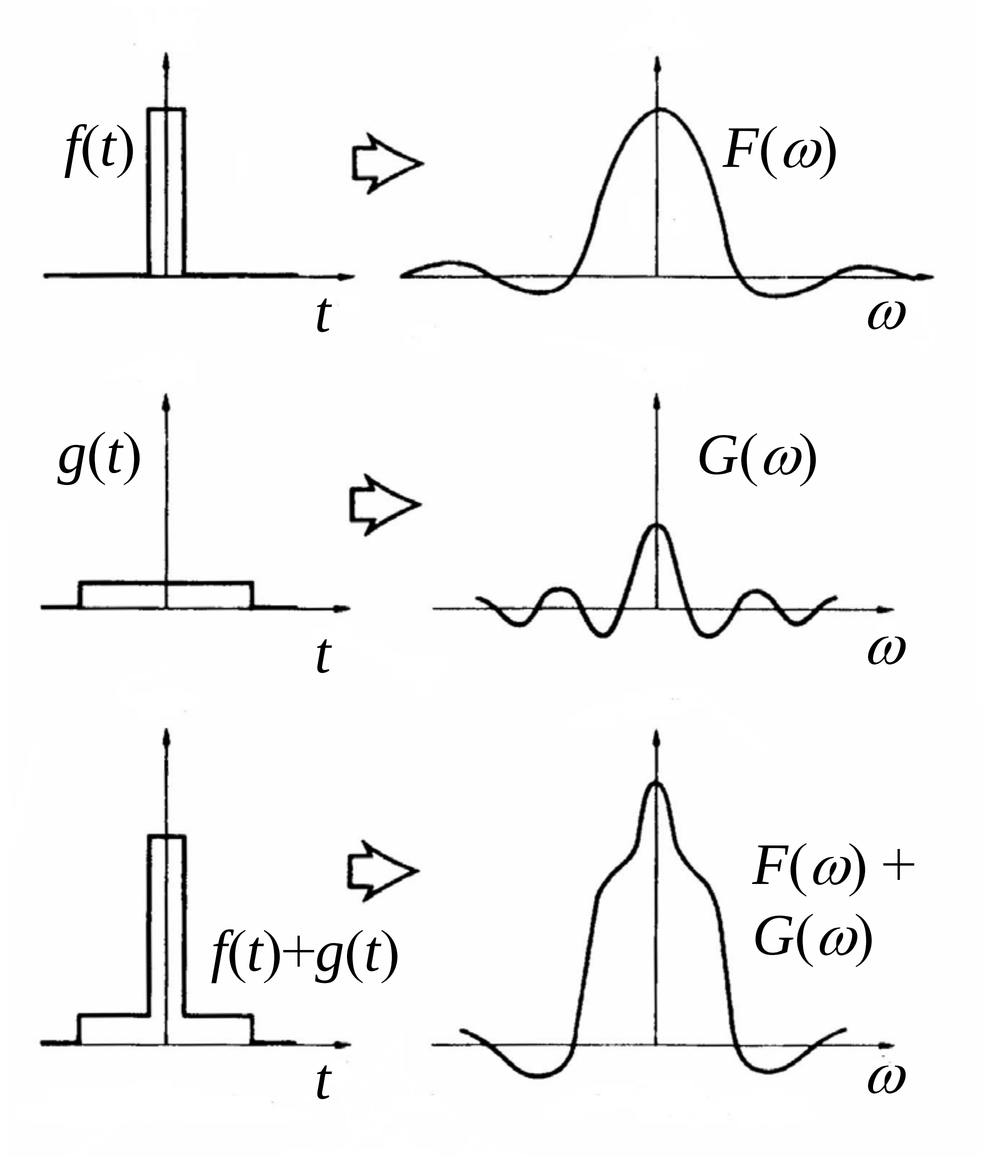
The shorter the pulse, the broader the spectrum!

This is the essence of the Uncertainty Principle!

The Fourier Transform of a sum of two functions

$$\boxed{\begin{aligned}&\mathsf{F}\ \{a\,f(t)+b\,g(t)\}=\\&a\mathsf{F}\ \{f(t)\}+b\mathsf{F}\ \{g(t)\}\end{aligned}}$$

Also, constants factor out.



Shift Theorem

The Fourier transform of a shifted function, $f(t-a)$:

$$\boxed{\mathcal{F}\left\{f(t-a)\right\} = \exp(-i\omega a)F(\omega)}$$

Proof :

$$\mathcal{F}\left\{f\left(t-a\right)\right\} = \int_{-\infty}^{\infty} f(t-a)\exp(-i\omega t)dt$$

Change variables : $u = t - a$

$$\int_{-\infty}^{\infty} f(u)\exp(-i\omega[u+a])du$$

$$= \exp(-i\omega a)\int_{-\infty}^{\infty} f(u)\exp(-i\omega u)du$$

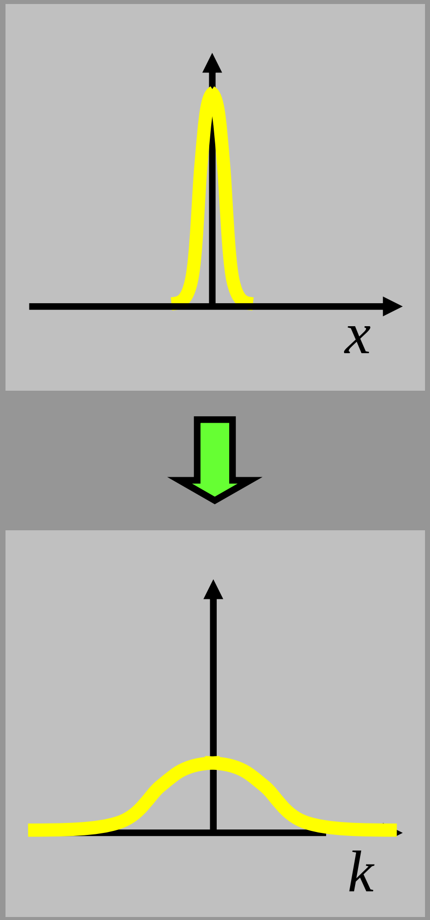
$$= \exp(-i\omega a)F(\omega)$$

Fourier Transform with respect to space

If $f(x)$ is a function of position,

$$\boxed{F(k) = \int_{-\infty}^{\infty} f(x)\exp(-ikx)\,dx}$$

$$\mathscr{F}\{f(x)\} \;=\; F(k)$$



We refer to k as the **spatial frequency**.

Everything we've said about Fourier transforms between the t and ω domains also applies to the x and k domains.

The 2D Fourier Transform

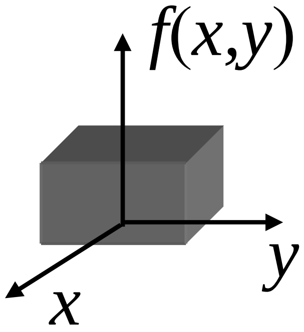
$$\mathscr{F}^{(2)}\{f(x,y)\} \;=\; F(k_x,k_y)$$

$$= \iint f(x,y)\,\exp[-i(k_x x+k_y y)]\;dx\;dy$$

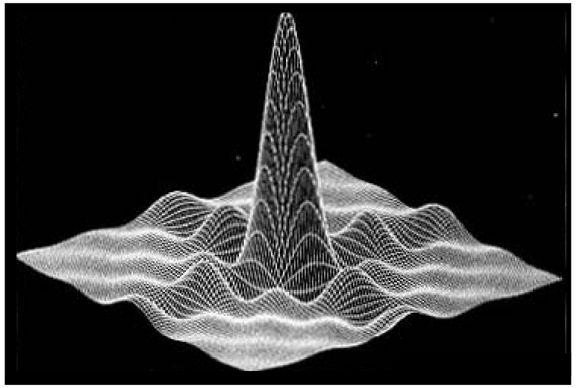
If $f(x,y) = f_x(x)\,f_y(y)$,

then the 2D FT splits into two 1D FT's.

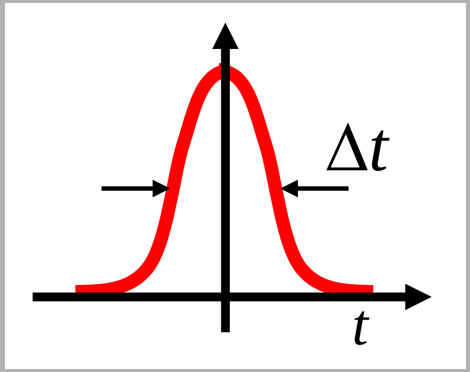
But this doesn’t always happen.



$\mathscr{F}^{(2)}\{f(x,y)\}$



The Pulse Width



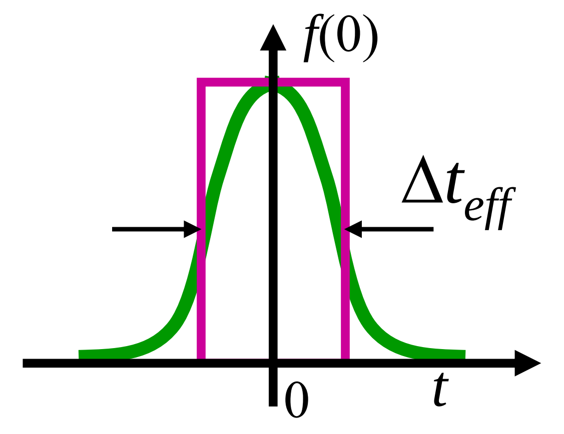
There are many definitions of the "width" or “length” of a wave or pulse.

The **effective width** is the width of a rectangle whose ***height*** and ***area*** are the same as those of the pulse.

Effective width ≡ Area / height:

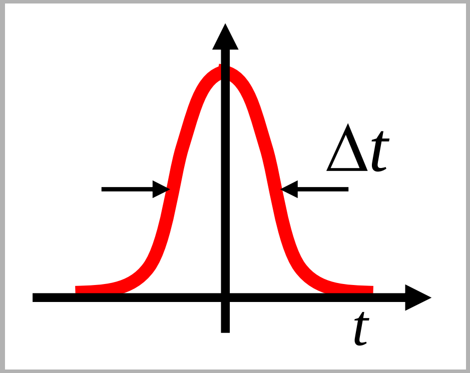
$$\Delta t_{eff} \equiv \frac{1}{f(0)} \int_{-\infty}^{\infty} \left| f(t) \right| dt$$

(Abs value is unnecessary for intensity.)



Advantage: It’s easy to understand.
Disadvantages: The Abs value is inconvenient.
We must integrate to ± ∞.

The rms pulse width



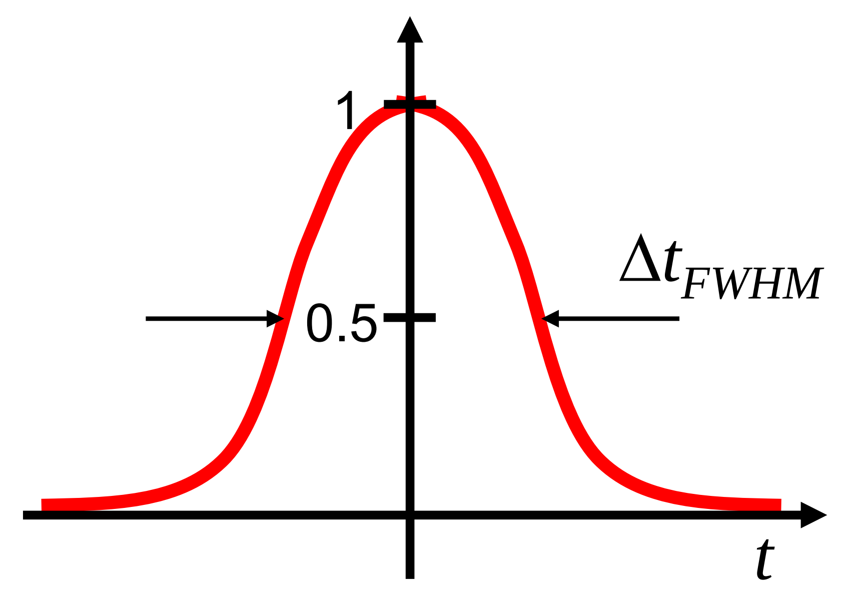
The **root-mean-squared width** or **rms width**:

$$\Delta t_{rms} \equiv \left[\frac{\int_{-\infty}^{\infty} t^2 f(t)\, dt}{\int_{-\infty}^{\infty} f(t)\, dt} \right]^{1/2}$$

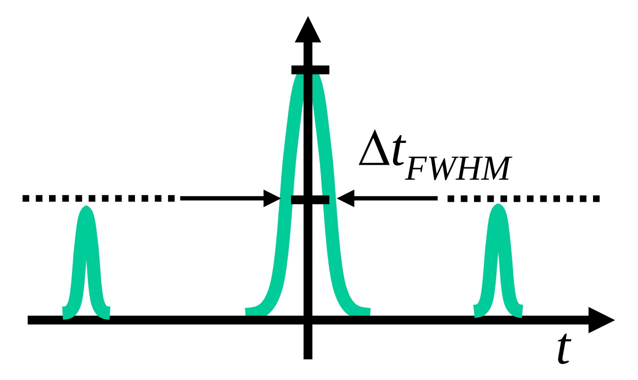
The rms width is the “second-order moment.”

Advantages: Integrals are often easy to do analytically.
Disadvantages: It weights wings even more heavily,
so it’s difficult to use for experiments, which can't scan to $\pm\infty$)

The Full-Width-Half-Maximum



Full-width-half-maximum is the distance between the half-maximum points.



Advantages: Experimentally easy.
Disadvantages: It ignores satellite pulses with heights < 49.99% of the peak!

Also: we can define these widths in terms of $f(t)$ or of its intensity, $|f(t)|^2$. Define *spectral* widths ($\Delta\omega$) similarly in the frequency domain ($t \rightarrow \omega$).

The Uncertainty Principle

The Uncertainty Principle says that the product of a function's widths in the time domain (Δt) and the frequency domain ($\Delta\omega$) has a minimum.

Define the widths assuming $f(t)$ and $F(\omega)$ peak at 0:

$$\Delta t \equiv \frac{1}{f(0)} \int_{-\infty}^{\infty} \left| f(t) \right| dt \qquad \Delta\omega \equiv \frac{1}{F(0)} \int_{-\infty}^{\infty} \left| F(\omega) \right| d\omega$$

$$\Delta t \geq \frac{1}{f(0)} \int_{-\infty}^{\infty} f(t)\, dt = \frac{1}{f(0)} \int_{-\infty}^{\infty} f(t) \exp(-i[0]t)\, dt = \frac{F(0)}{f(0)}$$

$$\Delta\omega \geq \frac{1}{F(0)} \int_{-\infty}^{\infty} F(\omega)\, d\omega = \frac{1}{F(0)} \int_{-\infty}^{\infty} F(\omega) \exp(i\omega[0])\, d\omega = \frac{2\pi f(0)}{F(0)}$$

Combining results:

$$\Delta\omega\, \Delta t \geq 2\pi \xcancel{\frac{f(0)\, F(0)}{F(0)\, f(0)}}$$

(Different definitions of the widths and the Fourier Transform yield different constants.)

or: $\boxed{\Delta\omega\, \Delta t \geq 2\pi}$ $\boxed{\Delta\nu\, \Delta t \geq 1}$

The Uncertainty Principle

For the rms width, $\Delta\omega \, \Delta t \geq ½$

There's an uncertainty relation for x and k: $\Delta k \, \Delta x \geq ½$