Levy Stable Distributions

https://en.wikipedia.org/wiki/Stable_distribution

Definition [edit]

A non-degenerate distribution is a stable distribution if it satisfies the following property:

> Let X_1 and X_2 be independent copies of a random variable X. Then X is said to be **stable** if for any constants $a > 0$ and $b > 0$ the random variable $aX_1 + bX_2$ has the same distribution as $cX + d$ for some constants $c > 0$ and d. The distribution is said to be *strictly stable* if this holds with $d = 0$.[6]

Since the normal distribution, the Cauchy distribution, and the Lévy distribution all have the above property, it follows that they are special cases of stable distributions.

Such distributions form a four-parameter family of continuous probability distributions parametrized by location and scale parameters μ and *c*, respectively, and two shape parameters β and α, roughly corresponding to measures of asymmetry and concentration, respectively (see the figures).

Although the probability density function for a general stable distribution cannot be written analytically, the general characteristic function can be. Any probability distribution is given by the Fourier transform of its characteristic function φ(t) by:

$$f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \varphi(t) e^{-ixt}\, dt$$

A random variable X is called stable if its characteristic function can be written as[6][7]

$$\varphi(t;\alpha,\beta,c,\mu) = \exp\left[\, it\mu - |ct|^{\alpha}\,(1 - i\beta\,\mathrm{sgn}(t)\Phi)\,\right]$$

where sgn(t) is just the sign of t and Φ is given by

$$\Phi = \tan(\pi\alpha/2)$$

for all α except α = 1 in which case:

$$\Phi = -\frac{2}{\pi}\log|t|.$$

$\mu \in \mathbf{R}$ is a shift parameter, $\beta \in [-1, 1]$, called the *skewness parameter*, is a measure of asymmetry. Notice that in this context the usual skewness is not well defined, as for $\alpha < 2$ the distribution does not admit 2nd or higher moments, and the usual skewness definition is the 3rd central moment.

Levy Stable Distributions (Cont.)

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The reason this gives a stable distribution is that the characteristic function for the sum of two random variables equals the product of the two corresponding characteristic functions. Adding two random variables from a stable distribution gives something with the same values of α and β, but possibly different values of μ and c.

Not every function is the characteristic function of a legitimate probability distribution (that is, one whose cumulative distribution function is real and goes from 0 to 1 without decreasing), but the characteristic functions given above will be legitimate so long as the parameters are in their ranges. The value of the characteristic function at some value t is the complex conjugate of its value at $-t$ as it should be so that the probability distribution function will be real.

In the simplest case $\beta = 0$, the characteristic function is just a stretched exponential function; the distribution is symmetric about μ and is referred to as a (Lévy) **symmetric alpha-stable distribution**, often abbreviated $S\alpha S$.

When $\alpha < 1$ and $\beta = 1$, the distribution is supported by $[\mu, \infty)$.

The parameter $c > 0$ is a scale factor which is a measure of the width of the distribution while α is the exponent or index of the distribution and specifies the asymptotic behavior of the distribution.

Levy Stable Distributions (Cont.)

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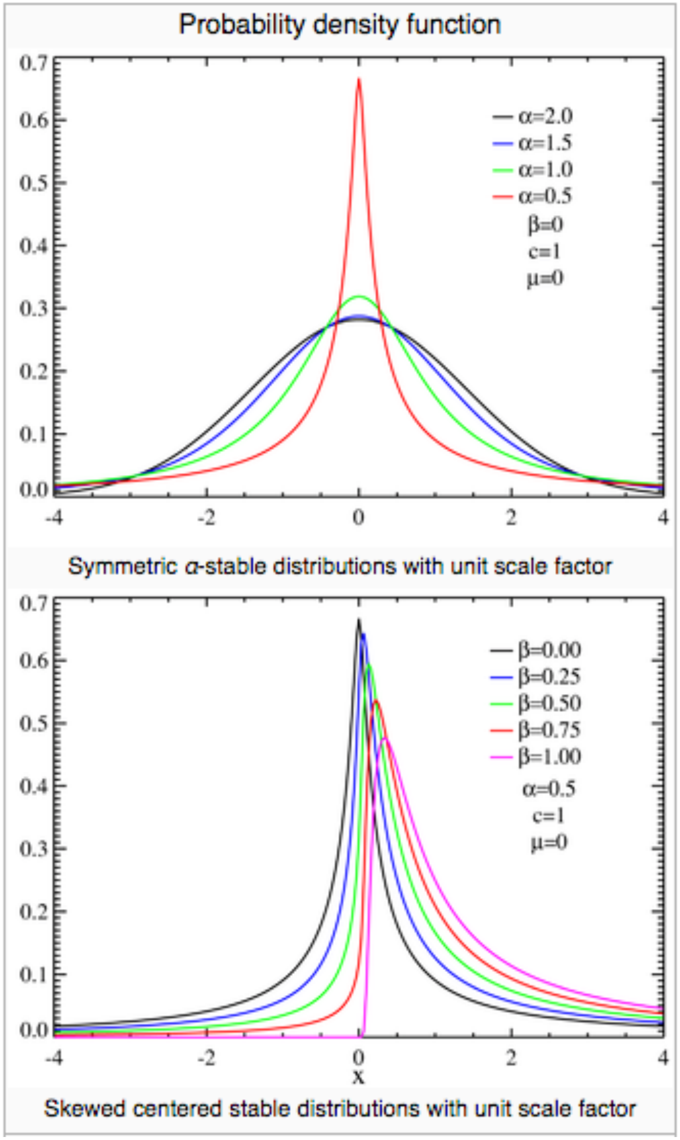

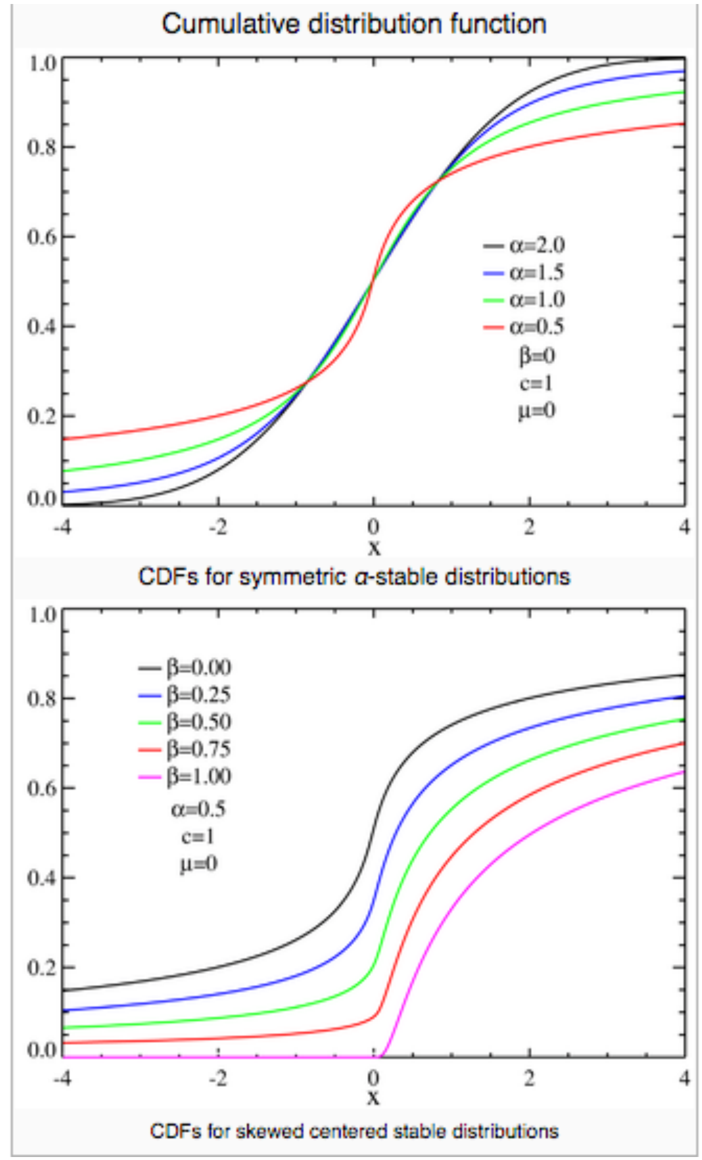

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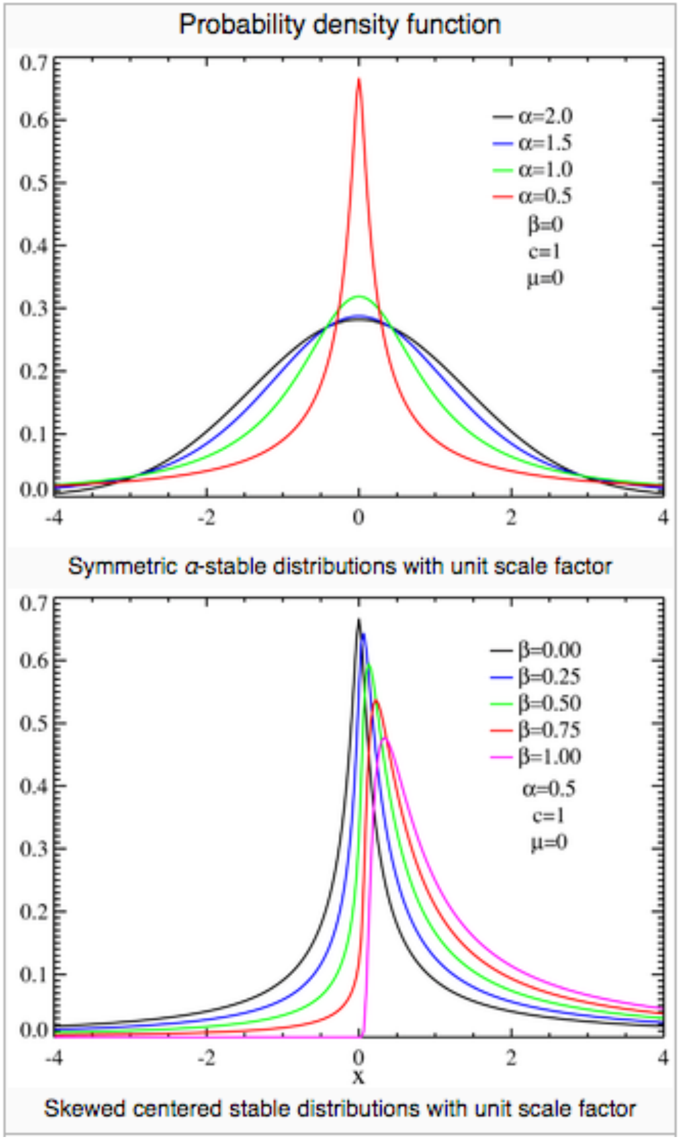
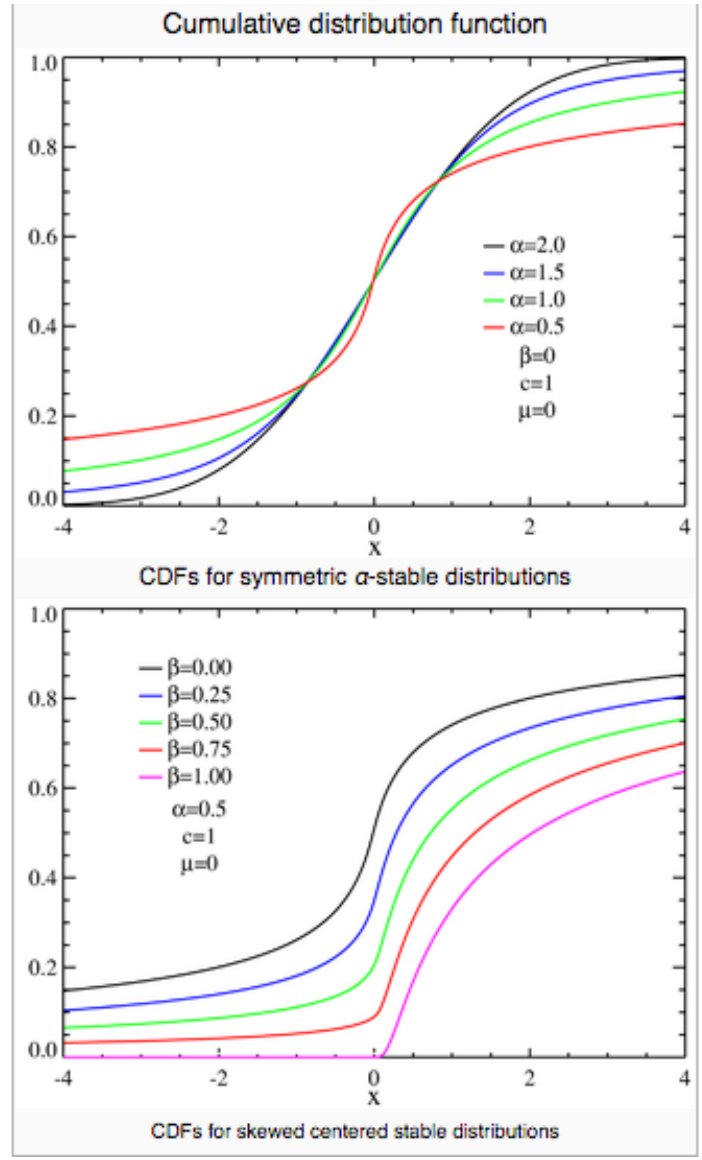


Symmetric α-stable distributions with unit scale factor

Skewed centered stable distributions with unit scale factor



CDFs for symmetric α-stable distributions

CDFs for skewed centered stable distributions

Levy Stable Distributions (Cont.)

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There is no general analytic solution for the form of $p(x)$.

There are, however three special cases which can be expressed in terms of *elementary functions* as can be seen by inspection of the *characteristic function*:

- For $\alpha = 2$ the distribution reduces to a **Gaussian** distribution with variance $\sigma^2 = 2c^2$ and mean μ; the skewness parameter β has no effect.
- For $\alpha = 1$ and $\beta = 0$ the distribution reduces to a **Cauchy** distribution with scale parameter c and shift parameter μ.
- For $\alpha = 1/2$ and $\beta = 1$ the distribution reduces to a **Lévy** distribution with scale parameter c and shift parameter μ.

Parameters	$\alpha \in (0, 2]$ — stability parameter $\beta \in [-1, 1]$ — skewness parameter (note that skewness is undefined) $c \in (0, \infty)$ — scale parameter $\mu \in (-\infty, \infty)$ — location parameter
Support	$x \in \mathbf{R}$, or $x \in [\mu, +\infty)$ if $\alpha < 1$ and $\beta = 1$, or $x \in (-\infty, \mu]$ if $\alpha < 1$ and $\beta = -1$
PDF	not analytically expressible, except for some parameter values
CDF	not analytically expressible, except for certain parameter values
Mean	μ when $\alpha > 1$, otherwise undefined
Median	μ when $\beta = 0$, otherwise not analytically expressible
Mode	μ when $\beta = 0$, otherwise not analytically expressible
Variance	$2c^2$ when $\alpha = 2$, otherwise infinite
Skewness	0 when $\alpha = 2$, otherwise undefined
Ex. kurtosis	0 when $\alpha = 2$, otherwise undefined
Entropy	not analytically expressible, except for certain parameter values
MGF	undefined
CF	$\exp\left[\, it\mu - \lvert c\,t\rvert^\alpha \left(1 - i\beta\,\mathrm{sgn}(t)\Phi\right)\right]$, where $\Phi = \begin{cases} \tan\frac{\pi\alpha}{2} & \text{if } \alpha \neq 1 \\ -\frac{2}{\pi}\log\lvert t\rvert & \text{if } \alpha = 1 \end{cases}$