Autocovariance:

$$C_{xx}(t_1, t_2) = E\left[(X(t_1) - \mu_{t_1})(X(t_2) - \mu_{t_2})\right]$$

$$= E\left[X(t_1)X(t_2)\right] - \mu_{t_1}\mu_{t_2}$$

Autocorrelation:

$$R_{xx}(t_1, t_2) = \frac{C_{xx}(t_1, t_2)}{\sigma_{t_1}\sigma_{t_2}}$$

If $\tau = t_2 - t_1$

$$R(\tau) = \frac{E\left[(X(t_1) - \mu)(X(t_2) - \mu)\right]}{\sigma^2}$$

$X(t_1) \rightarrow$ random variable at time $t_1$

$X(t_2) \rightarrow$ " " " " $t_2$

Measured by a quantity called <u>correlation time</u>, the time scale for the process

Consider

$$\int_0^\infty R(\tau)\, d\tau = \begin{cases} \text{finite} \\ \text{infinite} \\ \text{indeterminate} \end{cases}$$

(Indeterminate might correspond to, e.g.,

$\int_0^T R(\tau)\, d\tau$, $T \to \infty$, $= 0$ or $1$ depending on value of $T$ )

When integral is <u>finite</u>, we define

$$\tau_c = \int_0^\infty R(\tau)\, d\tau$$

↑ correlation time for process

<u>Examples</u>:

(a) $R(\tau) = \exp(-\tau/\tau_c)$ { exponential

$$\int_0^\infty \exp(-\tau/\tau_c)\, d\tau = \tau_c$$

For our Brownian particle,

$$S(f) = \int_{-\infty}^{\infty} \sigma^2 e^{-|\tau|/\tau_c} e^{-2i\pi f\tau}\, d\tau$$

$$\text{Power spectrum} = \frac{2\sigma^2\tau_c}{1+(2\pi f\tau_c)^2}$$

Use Cauchy residue theorem to find $S(f)$

$\Downarrow$

$|S(f)|^2$

When $f \ll 1/2\pi\tau_c$, power spectrum is basically frequency-independent. So for time windows $\Delta T \gg \tau_c$ stochastic process is basically white noise.

Integral of white noise is called a "Wiener Process", a non-stationary process characterized by a power-law power spectrum $S(f) \sim \frac{1}{f^2}$

"finger print" of short range stochastic process

## Long-Range Correlations

- Power law $S(f) \Rightarrow$ long-range correlations
- Such autocorrelation functions are seen in physics, biology, social science & economic systems

**So Consider**

$$S(f) = \frac{\text{const.}}{|f|^{\eta}} \quad \text{with} \quad 0 < \eta < 2$$

(i) $\eta = 0 \rightarrow$ white noise
(ii) $\eta = 2 \Rightarrow$ Wiener process

When $\eta \approx 1$, stochastic process with this spectral density is called $1/f$ noise or "flicker noise".

$1/f$ noise seen in diodes, transistors, traffic flow, etc.

Stochastic processes like this are non-stationary

## Markov processes.

Suppose a time scale $\tau_c$ exists and is defined.

Then for intervals $\Delta t > \tau_c$, we have

$$f(\underbrace{x_1, x_2 \ldots, x_{n-1}}_{\text{preceeding state}}; t_1, t_2 \ldots, t_{n-1} / \underbrace{x_n; t_n}_{\text{Current or next state}})$$

$$= f(x_{n-1}; t_{n-1} / x_n; t_n)$$

i.e., the pdf or state for the next value depends only the immediate last state

As a corollary:

$$f(x_1, x_2, x_3; t_1, t_2, t_3)$$

$$= f(x_1, t_1)\, f(x_1; t_1 / x_2; t_2)\, f(x_2; t_2 / x_3; t_3)$$

For a generalized Markov process, pdf depends on a finite number of preceeding states (not just 1)

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## Noises + Walks

Noise: Reverts to mean

Walks: Diffusion process fBm
fractional Brownian motion

Time Series: $\Delta t = t_2 - t_1$

$V(t)$: $\langle \Delta V \rangle^2 = \langle V(t_2) - V(t_1) \rangle^2$

For fBm

$\langle \Delta V \rangle^2 \sim \Delta t^{2H}$ H: Hausdorff dimension

Define a fractal Dimension D:

$D = 2 - H$

For Wiener process, $H = \frac{1}{2}$ (from a white noise)

equal power at all frequencies

Also $H > 1/2$ persistence
$H < 1/2$ anti persistence (pink noise)

Spectral exponent

$$\eta = 2H + 1$$