Levy Stochastic Processes

- Stable dist^n – does not change functional form as n changes (e.g. Gaussian)
- Another stable distribution is the Lorentzian dist^n:

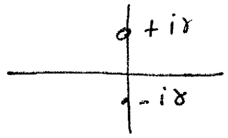
$$P(x) = \frac{\gamma}{\pi} \frac{1}{\gamma^2 + x^2} \quad \text{p.d.f.}$$

Fourier transform:

$$\varphi(q) \equiv \int_{-\infty}^{\infty} P(x)\, e^{iqx}\, dx$$

called the characteristic function of the stochastic process. For the Lorentz p.d.f., can use the Cauchy Residue theorem to evaluate:

$$\varphi(q) = e^{-\gamma|q|} \quad \text{"characteristic fn"}$$

[

Integral $= \frac{\gamma}{\pi}\int_{-\infty}^{\infty} \frac{e^{iqx}}{(x+i\gamma)(x-i\gamma)}\,dx$

$x>0$ close in u.h.p. $\Rightarrow 2\pi i \times \left(\frac{e^{-q\gamma}}{2i\gamma}\right) = e^{-q\gamma}$]

Convolution of 2 Functions – F.T. of convolution is product of F.T's of the 2 functions –

$$\Rightarrow \mathcal{F}[f(x) \otimes g(x)] = \mathcal{F}[f(x)]\,\mathcal{F}[g(x)] = F(q)G(q)$$

For iid random variables –

$$S_2 = x_1 + x_2$$

So the p.d.f. of the sum of two iid random variables is convolution of two pdfs of the random variable

$$P_2(s_2) = P(x_1) \otimes P(x_2)$$

see Wikipedia for a proof w/ Binomial

Assume Lorentzian distributed –

$$\varphi_2(q) = [\varphi(q)]^2$$

More generally,

$$P_n(S_n) = P(x_1) \otimes P(x_2) \otimes \cdots \otimes P_n(x_n)$$

$$S_n = x_1 + x_2 + \cdots + x_n \quad ; \quad \varphi_n(q) = [\varphi(q)]^n$$

For $S_2 = x_1 + x_2$, $\varphi_2(q) = e^{-2|q|\gamma}$

If we perform the inverse F.T.

$$P(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \varphi(q)\, e^{-iqx}\, dq$$

We obtain :

$$P_2(S_2) = \frac{2\gamma}{\pi}\,\frac{1}{4\gamma^2 + x^2}$$

See next pg

So the functional form is again Lorentzian. Thus the Lorentzian distⁿ is a stable distribution

For Gaussian random variables, analog is

$$P(x) = \frac{1}{\sqrt{2\pi}\,\sigma}\, e^{-x^2/2\gamma^2}$$

Characteristic function is

$$\varphi(q) = e^{-(\sigma^2/2)\,q^2} = e^{-\gamma q^2}$$

where $\gamma = \sigma^2/2$

$$\varphi_2(q) = e^{-2\gamma q^2}$$

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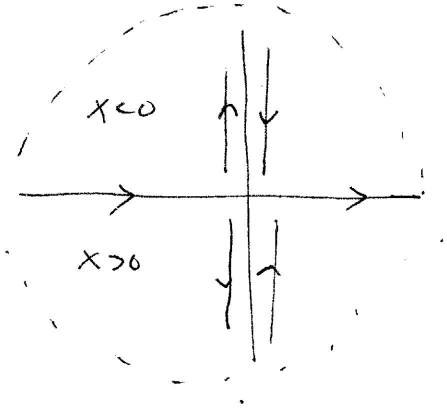
Inverse Fourier Transform of Lorentz dist^n



$$\frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-(2|q|\gamma + iqx)}\, dq$$

$$= \frac{1}{2\pi}\int_{-\infty}^{0} e^{-(iqx - 2q\gamma)}\, dq$$

$$+ \frac{1}{2\pi}\int_{0}^{\infty} e^{-(iqx + 2q\gamma)}\, dq$$

$$= \frac{1}{2\pi}\left[\frac{1}{2\gamma - ix} + \frac{-1}{-2\gamma - ix}\right] = \frac{2\gamma}{\pi}\,\frac{1}{4\gamma^2 + x^2}$$

By performing the inverse Fourier tsfm we obtain

$$P_2(S_2) = \frac{1}{\sqrt{8\pi\gamma}} e^{-x^2/8\gamma}$$

So the Gaussian distⁿ is also a stable distⁿ.

Rewriting, we have

$$P_2(S_2) = \frac{1}{\sqrt{2\pi}\,(\sqrt{2}\,\sigma)} e^{-x^2/2(\sqrt{2}\sigma)^2}$$

we find $\sigma_2 = \sqrt{2}\,\sigma$

So we now have found 2 stable stochastic processes — Gaussian & Lorentzian.

— Characteristic functions of both processes are

$$\varphi(q) = e^{-\gamma |q|^{\alpha}}$$

where $\alpha = 1$ for Lorentzian and $\alpha = 2$ for the Gaussian

Statistics of Price Changes

Note that Mandelbrot was the first in modern times to call attention to power-law price changes (Cotton prices ~1960's)

- In closed systems, infinite variance makes no sense (thermodynamics $\Rightarrow T \to \infty$)

- Also, power law dist^ns lack a characteristic scale & statisticians have never liked them for that reason

- Example is the St Petersburg paradox, introduced by N Bernoulli as a game in early 1700's (look it up!) and written about by D Bernoulli later

introduced notion of utility functions

The St Petersburg Paradox

- A banker flips a coin n+1 times.
- Player wins 2^{n-1} coins if n "tails" occur before the first "head"

Outcomes in following chart:

n	coins won	prob.	expected winnings
1	1	$1/2$	$1 \times 1/2 = 1/2$
2	2	$1/4$	$2 \times 1/4 = 1/2$
3	4	$1/8$	$4 \times 1/8 = 1/2$
$\vdots$	$\vdots$	$\vdots$	$\vdots$
n	2^{n-1}	$1/2^n$	$1/2$

If you play the game $\underbrace{\text{over \& over}}_{\text{"rounds"}}$, you would expect to win

$$\frac{1}{2} + \frac{1}{2} + \ldots + \ldots \Rightarrow \infty$$

- Question is, how much would you be willing to pay, for each round, to play this game? > N.B. introduced idea of utility

- Since banker stands to lose an ∞ # of coins, he would want an infinite # of coins for an ∞ # of rounds

- Player disagrees, because he cannot expect to win an ∞ # of coins with prob 1 (he will not win 2 coins or fewer with prob 3/4, etc.)

- Basic problem is that there is no scale in this problem, & player & banker are trying to determine a scale

- Today we discuss power laws in finite systems, where the power law is truncated by finite system size or other effects

- But nonetheless, scaling is an important concept

Price Change Statistics $\{x_i\}$

Two statistical properties

i) pairwise independent

ii) identically distributed

i) is basically true for "long" time intervals (subject to definition)

ii) is demonstrably not true.

volatility σ is time-dependent

What we need is a limit theorem that is based on (i) but not (ii)

Bawley & Khintchine: Sum S_n of random variables $\{x_i\}$

Assume $\{x_i\}$ are "infinitesimal", e.g., there is no particular variable x_ξ that dominates the sum

— Then Khintchine theorem says that it is necessary & sufficient that $F_n(S)$, the limiting dist^n function, be infinitely divisible

⟹ Infinitely Divisible Random Processes:

1. IDRP y is infinitely divisible if, for every natural number k, it can be represented as an iid sum of k variables $\{x_i\}$

2. The dist^n fn $F(y)$ is infinitely divisible iff the characteristic function $\varphi(q)$ is, for every nat. number k, the k^{th} power of some fn $\varphi_k(q)$.

Formally, $\varphi(q) = [\varphi_k(q)]^k$ $\qquad (*)$

with $\varphi_k(0) = 1$, and $\varphi_k(q)$ is continuous.

Stable Processes

Normally distributed random variable $\{y\}$ is inf. div. because

$$\varphi(q) = \exp\left[i\mu q - \frac{\sigma^2}{2} q^2\right]$$

So a solution of $(*)$ is

$$\varphi_k(q) = \exp\left[\frac{i\mu q}{k} - \frac{\sigma^2}{2k} q^2\right]$$

A symmetric stable random variable is infinitely divisible:

In fact:

$$\varphi(\xi) = \exp\left[i\mu\xi - \gamma|\xi|^\alpha\right]$$

So $$\varphi_k(\xi) = \exp\left[\frac{i\mu\xi}{k} - \frac{\gamma}{k}|\xi|^\alpha\right]$$

Poisson process

$$P(m,\lambda) = e^{-\lambda}\frac{\lambda^m}{m!}, \quad m = 0, 1, \ldots, n$$

$$\varphi(\xi) = \exp\left[\lambda\left(e^{i\xi} - 1\right)\right]$$

So $$\varphi_k(\xi) = \exp\left[\frac{\lambda}{k}\left(e^{i\xi} - 1\right)\right]$$

Gamma Dist^n

$$P(x) = \frac{e^{-x}\, x^{\nu-1}}{\Gamma(\nu)}$$

For $x \geq 0$ and $0 < \nu < \infty$, the characteristic function is

$$\varphi(\xi) = (1 - i\xi)^{-\nu}$$

So: $\varphi_k(q) = (1 - iq)^{-\nu/k}$

<u>Uniformly Distributed Random Variables</u>

$$P(x) = \begin{cases} 0 & x < -l \\ 1/2l & -l \le x \le l \\ 0 & x > l \end{cases}$$

Then: $\varphi(q) = \dfrac{\sin ql}{ql}$

So the process is <u>not</u> infinitely divisible because k^{th} root does not exist.

Note that there is strong evidence that the pdf of price changes must progressively converge to an infinitely divisable pdf for <u>long time horizons</u>

will prove to be an important requirement

Khintchine theorem ensures that for

large values of n $\left(\sum_{i=1}^{n} x_i = S_n\right)$

→ Price change dist^n is well defined although one may see fat tails

→ Price change stochastic process $Z(t)$ at time t may be characterized by parameters & functional forms that are t-dependent

→ Khintchine thm says that distribution $P(z)$ is close to an infinitely divisible pdf, with closer convergence as $n \to \infty$

→ Even in presence of volatility fluctuations, we can model price changes in terms of newly defined iid variables

View of stable & Levy stable distributions as fixed pts (attractors) in a system of distributions