Random Walks

Topics
- statistical properties
- central limit theorem
- asymptotic convergence to an attractor

d=1 Discrete case:

Consider sum of indep. ident. dist^n (iid) variables:

$$S_n = x_1 + x_2 + \dots + x_n \quad ; \quad x_n = x(n\Delta t)$$

— might be position of a random walker in 1d. n is number of steps at each time $t_n = n\Delta t$

iid variables $\{x_i\}$ are characterized by moments (mean, var, skew, kurt...) that do not depend on i.

Simplest random walk:

Random steps of size $\pm s = x_i$

Moments ("Expectation") of x_i:

$Mean(x_i) = E\{x_i\} = 0$

$Var\{x_i^2\} = E\{x_i^2\} = s^2$

For this random walk,

$E\{x_i x_j\} = \delta_{ij}\, s^2$

Note: Expectation of a sum of iid variables is sum of Expectations

Then:

$$E\{(S_n)\} = \sum_{i=1}^{n} E\{x_i\} = 0$$

also:

$$E\{S_n^2\} = \sum_{i=1}^{n}\sum_{j=1}^{n} E\{x_i x_j\} = \sum_{i=1}^{n} E\{x_i^2\}$$
$$= n s^2$$

So the distance traveled in a random walk is $\sim \sqrt{n}$

Continuous limit:

Let $n \to \infty$ and $\Delta t \to 0 \ni: t = n\Delta t$ is finite!

So $E\{X^2(t)\} = n s^2 = \dfrac{s^2 t}{\Delta t}$

S^2 must also be $\propto \Delta t$, or $\underline{S^2 = D\Delta t}$. Thus
on 1 time step

For consistency, we must have:

$S^2 = D\Delta t$ so that $E\{X^2(t)\} = Dt$

This expression characterizes a <u>diffusive</u> <u>process</u>

This is also called a <u>Wiener</u> <u>Stochastic</u> process.

Often, the increments $\left(\begin{matrix} x(t) \\ = S \end{matrix}\right)$ are considered to be <u>Gaussian</u> distributed.

- This equivalence to a Gaussian $\underbrace{\text{r.w.}}_{}$ random walk only generally holds as $n \to \infty$, because then the C.L.T. applies.
- In general, the increments $x(t) = S$ are characterized by some general non-Gaussian p.d.f.

<u>Note</u>: If x_1 & x_2 are iid, and have prob density fns $p(x)$, then if $z = x_1 + x_2$,

$$p(z) = p_1(x_1) \otimes p_2(x_2) \quad \text{where } \otimes = \text{Convolution}$$

$$= \int_{-\infty}^{\infty} p_1(z-y)\, p_2(y)\, dy$$

Note that in fig 3.2 of Mantegna & Stanley, that pdf's look more Gaussian as # of convolved variables increases (C.L.T.)

If functional form of $P(S_n)$ = same as form $P(x_i)$, stochastic process is stable in the mathematical sense ("Levy stable dist^n")

Central Limit Theorem:

Let S_n be a random variable composed of the iid variables x_i:

$$S_n = \sum_{i=1}^{N} x_i$$

Also $E\{x_i\} = 0$, $E\{x_i^2\} = s_i^2$

and $\sigma_n^2 = E\{S_n^2\} = \sum_{i=1}^{n} s_i^2$

Theorem (C.L.T.)

Define $\epsilon > 0$, and $U_i \equiv \begin{cases} x_i & \text{for } |x_i| \le \epsilon \sigma_n \\ 0 & \text{otherwise} \end{cases}$ every $\epsilon > 0$

(Then $\Rightarrow \bar{U}_i$ is a truncated random variable)

Now let

$$\frac{1}{\sigma_n^2} \sum_{i=1}^{n} E\{u_i^2\} \to 1 \quad \text{as} \quad \sigma_n \to \infty$$

Then the C.L.T says that the variable

$$\tilde{S}_n \equiv \frac{S_n}{\sigma_n} = \frac{x_1 + x_2 + \cdots + x_n}{\sigma_n \sqrt{n}}$$

is characterized by a Gaussian p.d.f. with unit variance:

$$P_G(\tilde{S}_n) = \frac{1}{\sqrt{2\pi}} \exp\left(- \tilde{S}_n^2/2\right)$$

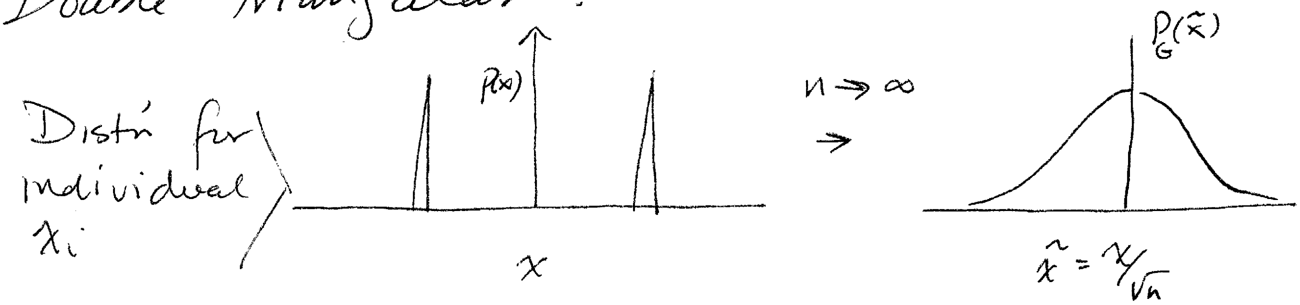
See Feller for a more rigorous proof.

However, one problem exists – there are distributions $P(x_i)$ for which

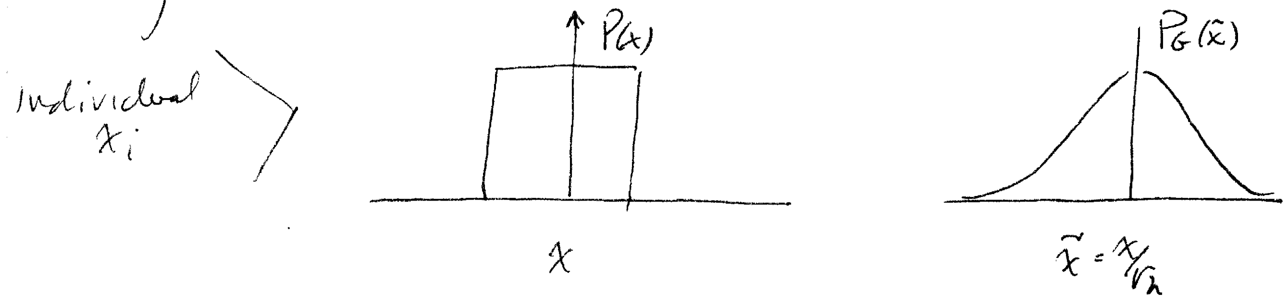
$E\{x_i^2\} \to \infty$. For these, the CLT does not apply. And unfortunately, stocks may be one of these cases

See the MS book for examples of convergence to the Gaussian shape for the double triangular + uniform dist$^{n's}$

Double triangular:



Uniform distn:



$$\tilde{P}(\tilde{x}) = P(\tilde{x})\sqrt{n} \to P_G(\tilde{x})$$

Speed of Convergence

→ Chebyshev considered this problem for a sum of iid variables x_i. He found that the difference between the function

$$F_n(s) = \int_{-\infty}^{s} \tilde{P}(\tilde{S}_n)\, d\tilde{S}_n$$

and the Gaussian depends on a series of polynomial functions. (see MS)

→ Alternately, we have the Berry-Esseen Theorems

Berry - Esséen Theorem 1

Let x_i have the property:

① $\begin{cases} a.\ E\{x_i\} = 0 \\ b.\ E\{x_i^2\} = \sigma^2 > 0, \quad \sigma^2 < \infty \\ c.\ E\{|x_i|^3\} \equiv \rho < \infty \end{cases}$ iid variables

Then if $\Phi(s)$ is a scaled Normal distⁿ, we have

$$|F_n(s) - \Phi(s)| \leq \frac{3\rho}{\sigma^3\sqrt{n}} \quad \text{as} \quad n \to \infty$$

So convergence is controlled by ratio of 3rd moment to cube of σ (sqrt 2nd moment)

Berry - Esseen Theorem 2:

Again assume properties 1a, 1b, but replace 1c by

$$E\{|x_i|^3\} \equiv r_i < \infty$$

$E\{x_i\} = 0$
$E\{x_i^2\} = \sigma_i^2$
$E\{|x_i|^3\} = r_i < \infty$
not iid

Define $S_n^2 = \sigma_1^2 + \sigma_2^2 + \dots + \sigma_n^2$

$\rho_n = r_1 + r_2 + \dots + r_n$

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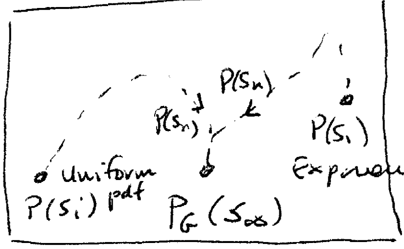
Fig 3.5 M&S



Then $\forall$ $n \to \infty$,

$$\left| F_n(S) - \Phi(S) \right| \leq \frac{6 P_n}{S_n^3}$$

Basin of Attraction

For nonlinear systems, iterations often lead to the concept of a basin of attraction, or more generally, an attractor. (Example of Lorenz attractor, Logistic map attractor, etc.). For example, it can be shown that neural networks have multiple basins of attraction that have been interpreted as memories

— Hopfield Neural Network

- $P_G(s)$ represents an attracting fixed pt. in the space of probability distributions having finite μ, σ.
- Note that other attractors exist as well!